Second Draft

Neutral Digits

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Abstract

Elementary number theorists are familiar with two major classes of digits n of a number base r. The divisor counting function enumerates all divisors $n \mid r$ and the Euler totient function counts all totatives $n \perp r$. A set of "neutral digits," neither divisors of nor coprime to r are investigated. Two types of neutral digits are proved to exist. Methods of construction and quantification of such digits are introduced.

Keywords: digit, number base, radix, divisor, totative, coprime, neutral digit, fundamental theorem of arithmetic, divisor counting function, Euler totient function, regular digit, regular number, semidivisor, semi-coprime number, semitotative.

Mathematics Subject Classification (MSC-2010) 11A99, 11N25.

1. Introduction

This work regards possible relationships between digits n of a number base r. This paper is the first of a series of papers and articles examining the nature and practical application of number bases. Thus, we will employ the variables mentioned rather than the usual variables seen in elementary number theory to convey the equivalent of digits and number bases.

A digit is a positive integer $0 < n \le r$, where the symbol "0" (i.e., zero) symbolizes congruence with r. Observe that in the decimal number 20, the digit zero in the rightmost place simply signifies that the quantity twenty is congruent with 10^1. When the digit 0 stands alone, the symbol "0" stands for actual zero. We will ignore the case where "0" connotes actual zero, since this produces complications. Thus in this paper, the digit $0 \equiv r$. [1] The digit $\omega = (r - 1)$ is the greatest and final digit of base r. Thus the "digit range" of base r is $\{0, \ldots, \omega\}$, includes all possible digits of r, and numbers r elements.

0	1	2	3	4	5	6	7	8	9

Figure 1.1. The range of digits n *of base* r *= 10.*

A number base or radix is a positive integer $r \ge 2$. [2] To be sure, a radix need not be confined to positive integers, but for the sake of this study, we'll focus on the case where r is strictly a positive integer ≥ 2.

Let's review some basic aspects of elementary number theory before exploring the relationship of digits and number bases. We will presume a knowledge of primes [3, 4, 5, 6], composites [7, 8, 9, 10, 11], and units [12, 13, 14]. Familiarity with the three relationships $n \mid r$ (n divides r evenly, or n is a divisor of r), $n \nmid r$ (n does not divide r evenly, or n is not a divisor of r), and $n \perp r$ (n is coprime to r, or n is relatively prime to r) is also important to this investigation. Note that in this paper, we will use the term "coprime" adjectivally when discussing a digit of r, and use the noun "totative" for a digit which is coprime to r. We will use the term "factor" at times exclusively for divisors of digits when contrast needs to be made between factors of a digit and divisors of r. Numbers considered in this paper are generally positive integers, except the number $r^{½}$, the square root of r.

Some observations regarding prime numbers should be kept in mind. Consider an arbitrary prime p and a larger arbitrary positive integer r. The prime digit p must either be a divisor of r or coprime to r [15]. A prime base p is the product of its trivial divisors $\{1, p\}$ and is coprime to all numbers $n < p$, including $n = 1$. By these observations, two things are clear:

1. Prime digits p must either be divisors or totatives of an arbitrary number base r. We need not examine prime digits to determine if they are neutral.
2. No neutral digit can exist for a prime base p. We need not consider the existence of neutral digits for prime bases; they do not exist.

Therefore, neutral digits must be composite and must exist only in composite bases.

Three formulæ prove essential in this investigation. The first is the standard form of prime decomposition denoting the distinct prime divisors of r and their exponents α [16]. Let i and k be positive integers with $1 \le i \le k$. Let p_k be the k-th distinct prime divisor of r, and the integer $\alpha_i \ge 1$ be the multiplicity [17] of the distinct prime divisor p_i in r.

$$r = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k} \tag{1.1}$$

$$(\alpha_1 > 0, \alpha_2 > 0, \ldots, \alpha_k > 0, p_1 < p_2 < \ldots < p_k)$$

The second key formula relates a divisor d with a divisor's complement d', both the divisor and its complement being positive integers. [18]

$$r = d \cdot d' \tag{1.2}$$

Let the integer $1 \le \delta_i \le \alpha_i$ be the multiplicity of each of the base r's distinct prime divisors p_i in the divisor d. Each divisor d thus has a standard form prime decomposition

$$d = p_1^{\delta_1} p_2^{\delta_2} \ldots p_r^{\delta_r} \tag{1.3}$$

$$(p_1 < p_2 < \ldots < p_k)$$

Formulæ specific to certain aspects of this paper will be introduced later.

2. The Existence of Neutral Digits.

Let D be the set of divisors of r, and T be the set of totatives of r. Let $d \in D$ be a positive integer $d \mid r$ and $t \in T$ be a positive integer $t \perp r$. There are two well-known functions which count the number of divisors in the set D and totatives in the set T. Through examination of the behavior of these functions, we can see that there exist integers $0 < s < r$ for some values of r that are neither divisors of nor coprime to r. In the case of r as a number base, these numbers s are digits, because these are less than or equal to r, yet still positive. Thus, we will refer to such numbers s as "neutral digits".

Figures 1.2 and 1.3 show the digits $d \mid r$ and $t \perp r$ respectively, given $r = 10$.

0	1	2	3	4	5	6	7	8	9

Figure 1.2. The divisors d *of base* r *= 10, shown in red.*

0	1	2	3	4	5	6	7	8	9

Figure 1.3. The totatives t *of base* r *= 10, shown in blue.*

THE DIVISOR COUNTING FUNCTION.

The divisor counting function, $\sigma_0(r)$, counts the number of positive divisors of the integer r [19, 20]. Let p denote a prime number. Then $\sigma_0(p) = 2$, since all prime numbers possess the trivial divisors $\{1, p\}$ [21]. Because σ_0 is a multiplicative function [22], we can produce σ_0 for any number r given its prime decomposition in standard form given by formula (1.1). Let i and k be positive integers with $1 \le i \le k$.

$$(2.1) \quad \sigma_0(r) = \prod_{i=1}^{k} (\alpha_i + 1) = (\alpha_1 + 1)(\alpha_2 + 1)\dots(\alpha_k + 1).$$

The divisor counting function counts an increasingly small proportion of the digits of base r as r increases. Observe that every distinct prime divisor p_i contributes a factor $(\alpha_i + 1)$ to formula (2.1), regardless of the magnitude of p_i itself. The value of each factor depends on the multiplicity α_i of each prime divisor p_i. We will show that there is at least some "room" for other types of digits by examining the behavior of the divisor counting function $\sigma_0(r)$.

Theorem 2.1. Let the positive integer $r \ge 2$. The divisor counting function $\sigma_0(r) < r$ for all values of $r > 2$, and $\sigma_0(r) = r$ for $r = 2$.

Lemma 2.1.1. Let the integer $i = 1$ and the multiplicity $\alpha_i = 1$ for all p_i. The number of divisors for all prime numbers is 2.

Proof. We can rewrite formula (2.1) as

$$(2.2) \quad \sigma_0(r_1) = (\alpha_1 + 1) = (1 + 1) = 2$$

The two divisors are $\{1, r\}$, the trivial divisors, which are divisors of every integer. ▲

Lemma 2.1.2. Let i and k be positive integers with $1 \le i \le k$. Let the multiplicity $\alpha_i = 1$ for all p_i. The quantity of divisors for all squarefree [23] products $p_1 \cdot p_2 \cdot \dots \cdot p_k$ is 2^k.

Proof. We can rewrite formula (3) as

$$(2.3) \quad \sigma_0(r_i) = \prod_{i=1}^{k} (\alpha_i + 1) = 2^k$$

Each prime divisor p_i contributes the factor $(\alpha_i + 1)$ to the equation. Since all multiplicities $\alpha_i = 1$, all the factors $(\alpha_i + 1) = 2$. Lemma 2.1.1 shows that a prime r yields 2 divisors. Each additional distinct prime divisor contributes a factor 2 to formula (2.3). ▲

Lemma 2.1.3. The quantity of divisors for all primorials $r\# = p_1 \cdot p_2 \cdot \dots \cdot p_k$ is 2^k.

The primorial $r\#$ is a special case of Lemma 2.1.2. Let $j \ge 1$ be a positive integer. Let π be a prime number such that π_j is the j-th prime, with $\pi_1 = 2$. If the prime decomposition of the primorial $r\#$ is in standard form, then $i = j$. The least prime divisor $p_1 = \pi_1 = 2$. Further, each prime divisor p_i is the j-th prime number π_j.

$$(2.4) \quad p_i = \pi_j$$

Since $r\#$ is squarefree, we can use formula (2.3) to compute its divisor counting function. A primorial $r\#$ and a squarefree r differ in two possible ways. The primorial will have its least prime divisor $p_1 = 2$, while the squarefree r may have an arbitrary prime as its least prime divisor. The primorial $r\#$ will have its i-th prime divisor $p_i = \pi_j$, the j-th prime number, and $i = j$ for all prime divisors of $r\#$. The prime divisors p_i of a squarefree r may be arbitrary, with i not necessarily matching j. Thus a squarefree r is a primorial $r\#$ if and only if the prime decomposition is in standard form and $i = j$ for all p_i. An example of a primorial $r\#$ is $30 = 2 \cdot 3 \cdot 5$. Each $p_i = \pi_j$ and $p_1 = 2$. An example of a squarefree r which is not a primorial is $165 = 3 \cdot 5 \cdot 11$. The least prime divisor $p_1 \ne 2$ and all prime divisors $p_i \ne \pi_j$. The primorial $r\#$ is a special case of a squarefree r. Additionally, the primorial $r_k\#$ is the smallest squarefree r_k. ▲

Proof of Theorem 2.1. Theorem 2.1 can be broken into two cases, with one case consisting of two situations.

CASE 1. The divisor counting function $\sigma_0(r) = r$ for $r = 2$. The number 2 is prime, thus by Lemma 2.1.1, $\sigma_0(r) = 2$. The divisor counting function $\sigma_0(r) = r$, since $r = 2$. Observe that $r = 2$ is the only prime for which $\sigma_0(r) = r$, since 2 is the smallest prime. Thus $\sigma_0(r) < r$ for all prime values of $r > 2$.

CASE 2. The divisor counting function $\sigma_0(r) < r$ for all values of $r > 2$. We need to examine two extreme situations of this case. Let $k > 1$ with $1 < i \le k$, since Case 1 covers prime values of r.

SITUATION 1. Let's consider a composite squarefree $r = p_1 \cdot p_2 \cdot \dots \cdot p_k^a$ and $r' = r \cdot p_k = p_1 \cdot p_2 \cdot \dots \cdot p_k^{(a+1)}$. All the prime divisor exponents are equal in both r and r' except the multiplicities of p_k. Further, $\gcd(r, r') = r$, and p_k is a common divisor of r and r'. Let's assume $a = 1$. Since $k > 1$ and each p_i is a distinct prime factor, $p_k > 2$. The ratio of the divisor counting functions $\sigma_0(r') / \sigma_0(r) = (a + 1 + 1) / (a + 1) = 3/2$, while the ratio $r'/r = p_k > 2$. In the situation where $a = 1$, the ratio r'/r is greater than that of $\sigma_0(r') / \sigma_0(r)$. Through mathematical induction, when the multiplicity of the greatest prime divisor is increased by one, r grows faster than $\sigma_0(r)$.

Situation 2. Now let's consider a similar situation where the multiplicity of the least prime divisor is manipulated. Let $p_1 = 2$. Consider a composite squarefree $r = p_1^a \cdot p_2 \cdot \ldots \cdot p_k$ and $r' = r \cdot p_k = p_1^{(a+1)} \cdot p_2 \cdot \ldots \cdot p_k^{(a+1)}$. Again all the prime divisor exponents are equal in both r and r' except the multiplicities of p_k. Further, $\gcd(r, r') = r$, and p_1 is a common divisor of r and r'. Since $p_1 = 2$, the ratios $p_1^{(a+1)}/p_1^a = r'/r = p_1 = 2$. The ratio of the divisor counting functions $\sigma_0(r') / \sigma_0(r) = (b+1) / (a+1) = 3/2$. In the situation where the multiplicity of the smallest possible prime divisor $a = 1$, the ratio r'/r is greater than that of $\sigma_0(r') / \sigma_0(r)$. Through mathematical induction, when the multiplicity of the greatest prime divisor is increased by one, r grows faster than $\sigma_0(r)$.

Clearly $\sigma_0(r) < r$ for all composite values of r, since $r \cdot p > \sigma_0(r \cdot p)$, regardless of the magnitude of p. Thus, for all $r > 2$, we have $\sigma_0(r) < r$, and $\sigma_0(r) = r$ if and only if $r = 2$. ▲

The Euler totient function.

The Euler totient function $\phi(r)$ counts the positive integers $0 < n \le r$ that are coprime to r [24, 25]. In other words, the Euler totient function counts totatives of r. The Euler totient function $\phi(p)$ for a prime p is:

(2.5) $$\phi(p) = p - 1,$$

since all digits except the digit $0 \equiv p$ itself are coprime to p [26]. Through formula (2.5), we observe that $\phi(p) \to p$ as $p \to \infty$. Consider a positive composite integer $r > 2$. The prime decomposition of r in standard form is given by formula (1.1). Let i and k be positive integers with $1 \le i \le k$. The Euler totient function [27, 28] for all composite numbers r is

(2.6) $$\phi(r) = r \cdot \prod_{i=1}^{k} (1 - 1/p_i)$$
$$= r(1 - 1/p_1)(1 - 1/p_2)\ldots(1 - 1/p_k).$$

From formula (2.6) we can see that each distinct prime divisor p_i contributes one factor $(1 - 1/p_i)$. The factor is reliant on the magnitude of p_i but the multiplicity of p_i is immaterial in each factor.

If we consider the totient ratio

(2.7) $$\phi(r)/r = \prod_{i=1}^{k} (1 - 1/p_i)$$

the base r and the magnitude of its digit range are scaled to 1, and we can observe the effects of distinct prime factors on the proportion of totatives in base r. (See Figure 2.1.) We only need to consider squarefree versions ρ of bases r such that

(2.8) $$r = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k} \longrightarrow \rho = p_1 p_2 \ldots p_k$$

e.g., $12 = 2^2 \cdot 3 \longrightarrow 6 = 2 \cdot 3$ and $360 = 2^3 \cdot 3^2 \cdot 5 \longrightarrow 30 = 2 \cdot 3 \cdot 5$.

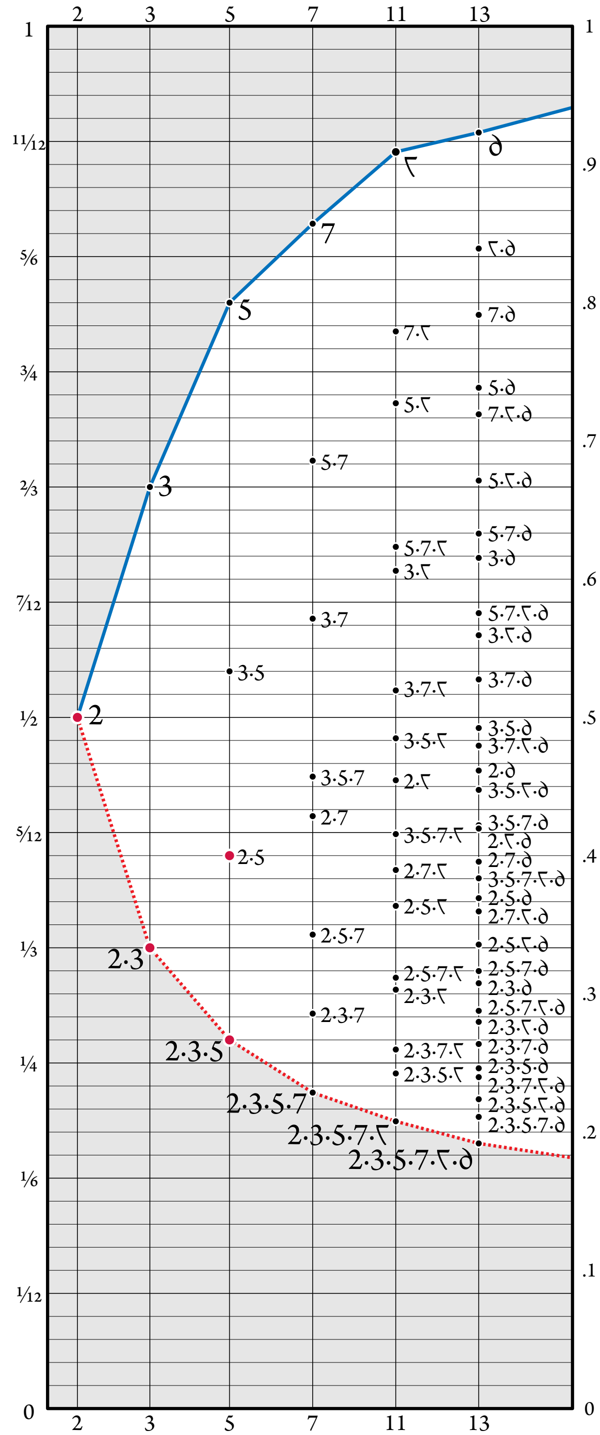


Figure 2.1. A plot with $\phi(r)/r$ on the vertical axis versus the maximum distinct prime divisor p_{max} on the horizontal axis. (The horizontal axis is not to scale.) The p_{max}-smooth numbers lie along a vertical line at each value of p_{max}. The boundary of minimum values of $\phi(r)/r$ defined by primorials is indicated by a broken red line. The boundary of maximum values of $\phi(r)/r$ defined by primes is shown in blue. All other composite numbers r that have p_{max} as the maximum distinct prime divisor inhabit the region between the boundaries. See Figure A1 in the Appendix for a plot of $\phi(r)/r$ vs. p_{max} to scale with wider scope.

Theorem 2.2. Let $p_1 = 2$, the smallest possible distinct prime divisor of an arbitrary base *r*. Let the positive integer $k > 1$ and p_k be the largest prime divisor of *r*. Consider the primorial $r\# = p_1 p_2 \cdots p_k$. The totient ratio $\phi(r)/r$ must be

(2.9) $$\phi(r\#)/r\# \le \phi(r)/r \le \phi(p_k)/p_k.$$

Let's consider two cases: $\phi(r)/r < \phi(p_k)/p_k$ and $\phi(r)/r > \phi(r\#)/r\#$.

Lemma 2.2.1. Let p_1 be the smallest distinct prime divisor of base *r*. Let the positive integer $k > 1$ and p_k be the largest prime divisor of *r*. Bases which are large, single primes p_k possess a totative ratio $\phi(p_k)/p_k \ge \phi(r)/r$ for bases *r* whose largest distinct prime divisor is p_k.

Proof. Let *i* and *j* be positive integers with $1 \le i \le k$ and *j* is arbitrary. Let π_j be the *j*-th prime number. Suppose $p_1 = \pi_j$ and $p_i = \pi_{(j+i-1)}$. Consider $r = p_1 p_2 \ldots p_k$. The largest distinct prime factor p_k reduces the ratio $\phi(r)/r$ in formula (2.7) by $(1 - 1/p_k)$. Every smaller prime divisor p_i reduces the ratio somewhat more than p_k, since $(1 - 1/p_i) < (1 - 1/p_k)$. Since p_1 is the smallest prime divisor of *r*, it has the greatest effect on the ratio $\phi(r)/r$. Thus $\phi(p_k)/p_k > \phi(r)/r$ for composite bases *r* whose greatest distinct prime divisor is p_k. Suppose $p_1 = 2$. Then $\phi(p_1)/p_1 = ½$. Every prime number p_i is larger than 2 and every prime number $p_{i+1} > p_i$, thus $(1 - 1/p_{i+1}) > (1 - 1/p_i) > ½$. Thus $\phi(p)/p \to 1$ as $p \to \infty$. The larger the prime, the greater the proportion of its totatives among its digits. Because $\phi(p) = p - 1$, $\phi(p)/p$ will never reach 1, as there will always be 1 non-totative digit, *p*. It is clear that if $r = p_k$ (i.e., *r* is a prime p_k) then $\phi(p_k)/p_k = \phi(r)/r$. Therefore, $\phi(p_k)/p_k \ge \phi(r)/r$ for bases *r* that have p_k as the largest distinct prime divisor. ▲

Lemma 2.2.2. Let *i* and *k* be positive integers with $1 \le i \le k$. Let $p_1 = 2$, and p_i be the *i*-th prime number. Primorial bases $r\#$ possess a totative ratio $\phi(r\#)/r\# \le \phi(r)/r$ for arbitrary composite bases *r* whose largest distinct prime divisor is p_k.

Proof. There must be *k* distinct prime divisors of the primorial $r\#$ by definition. This implies that there are *k* factors $(1 - 1/p_i)$, each of which reduce the totient ratio somewhat, and none of which maintain the ratio. (In other words, the factors each must be less than 1). Any arbitrary $r \ne r\#$ must contain one or more primes p_i, and all the primes p_i that divide *r* must also divide $r\#$. Thus all the effects on $\phi(r)/r$ of every $p_i \mid r$ through the factor $(1 - 1/p_i)$ also occur on $\phi(r\#)/r\#$. There must be at least one p_i which does not affect $\phi(r)/r$ otherwise *r* must equal $r\#$, as *r* and $r\#$ would then have the same prime decomposition. Therefore any $r \ne r\#$ must have $\phi(r)/r \ge \phi(r\#)/r\#$. ▲

Proof of Theorem 2.2. From Lemma 2.2.1, we proved that $\phi(p_k)/p_k > \phi(r)/r$, while Lemma 2.2.2 supplies a proof that $\phi(r)/r > \phi(r\#)/r\#$. Then we have the relationship described in formula (11) for squarefree bases *r*. It is evident from formula (2.6) that the multiplicity *a* of the distinct prime divisors *p* of base *r* are immaterial in computing the Euler totient function, thus admitting the substitution of all bases *r* with a squarefree version r' through formula (2.7). Thus formula (2.9) applies to all positive integers $r \ge 2$. ▲

From formula (2.9) and the fact that prime bases *p* cannot possess neutral digits, since $\sigma_0(p) = 2 \wedge \phi(p) = p - 1$, we can deduce that there may be room for neutral digits in composite bases. We need to review the nature of the digit 1 and produce a neutral digit counting function so that we can better prove the existence of neutral digits in composite bases.

The Unit.

It is appropriate to acknowledge observations regarding the digit 1. The digit 1 of all bases *r* is special and has a complement in the digit $\omega = (r - 1)$. The digit 1 is a totative of every base *r* since $\gcd(1, r) = 1$. The digit $\omega = (r - 1)$ is also a totative of every base *r*, since a positive integer $m \perp (m + 1)$ [29]. If $r = 2$, then 1 and ω refer to the same totative 1. Thus $\{1, r - 1\}$ are totatives for all $r \ge 2$.

Recall that for all bases $r \ge 2$, the digit $1 \mid r$. Therefore, the digit $1 \mid r \wedge 1 \perp r$. This implies that the digit 1 is counted by both $\phi(r)$ and $\sigma_0(r)$. Let's prove that the digit 1 is unique in its status as both a divisor and a totative of *r*. This would establish that $\phi(r)$ and $\sigma_0(r)$ are mutually exclusive except in the case of the digit 1.

Theorem 2.3 The digit 1 is the only positive integer $0 < n \le r$ such that $1 \mid r \wedge 1 \perp r$.

Proof. Let a positive integer $1 \le d < r$ with $d \mid r$. Then $\gcd(d, r) = d$. The equation $\gcd(d, r) = 1$ is true if and only if $d = 1$. Thus the cases where $d \mid r$ and $d \perp r$ are mutually exclusive except in the case of $d = 1$. The number 1 is thus uniquely both a divisor of and coprime to *r*, since 1 divides all *r* evenly and $\gcd(1, r) = 1$. ▲

The following formula defines a "countable digits function" $\varkappa(r)$ that quantifies any digit counted by $\sigma_0(p) \vee \phi(p)$, effectively yielding the population of $D \cup T$:

(2.10) $$\varkappa(r) = \phi(r) + \sigma_0(r) - 1$$

which avoids counting the digit 1 twice. The digit 1 is referred to as "a unit" [30] since it is neither prime nor composite. The digit 1 may well be classified as a unit in terms of relationship with *r*, since for all bases $r \ge 2$, $1 \mid r \wedge 1 \perp r$.

Figure 2.2 illustrates the set of decimal countable digits $\mathcal{K} = \{0, 1, 2, 3, 5, 7, 9\}$. Figure 2.3 shows the sets of countable digits $\mathcal{K}$ of bases $2 \le r \le 16$. From these limited charts, it is evident that there are some digits *n* in some bases *r* for which neither $n \mid r$ nor $n \perp r$ is true. These digits are neutral digits. Figure 2.4 illustrates the set *S* of decimal neutral digits. Figure 2.5 shows all neutral digits in number bases between 2 and 16 inclusive.

A Test for Neutral Digits.

We can use $\gcd(n, r)$ to test the relationship of the digit *n* to the base *r*. It is well known that $n \perp r$ if $\gcd(n, r) = 1$. It follows from the definition of $n \mid r$ that $\gcd(n, r) = n$. Since neither $n \mid r$ nor $n \perp r$ is true, a neutral digit *n* will have:

(2.11) $$1 < \gcd(n, r) < n.$$

3. The Neutral Digit Counting Function.

The divisor counting function $\sigma_0(r)$ and the Euler totient function $\phi(r)$ can be used to count neutral digits.

Theorem 3.1. Let the positive integer $r \geq 2$ be a number base. The neutral digit function $\nu(r)$ is given by the following equation:

$$
\begin{aligned}
(3.1) \qquad \nu(r) &= r - \varkappa(r) \\
&= r - [\phi(r) + \sigma_0(r) - 1] \\
&= r - \left(\prod_{i=1}^{k} (\alpha_i + 1) + r \cdot \prod_{i=1}^{k} (1 - {}^1\!/_{p_i}) - 1 \right)
\end{aligned}
$$

Proof. The digit range of r must equal r, thus any catenation of functions counting digits of r must be less than r. The divisor counting function and Euler totient function yield the quantity of divisors and totatives of r, respectively. We must subtract 1 from the sum $\phi(r) + \sigma_0(r)$ since $1 \mid r \ \wedge\ 1 \perp r$. We will tackle the proof using two cases.

Case 1. Let p be an arbitrary positive prime, and suppose $r = p$. No neutral digit can exist for a prime base p. Given Lemma 2.1.1 and formula (2.5), we have the following:

$$
\begin{aligned}
(3.2) \qquad p &= \phi(p) + \sigma_0(p) - 1 \\
&= (p - 1) + 2 - 1 \\
&= p - 1 + 1 \\
&= p
\end{aligned}
$$

Case 2. Suppose $r > 2$ and is an arbitrary composite number. Theorem 2.1 shows that $\sigma_0(r) < r$. Formula (2.9) of Theorem 2.2 shows that the prime base p will have the greatest ratio of totatives to digits, and that the presence of smaller primes in the prime decomposition of an arbitrary composite base r will admit some room for neutral digits in the range of digits of some composite number bases r.

In either case, $\phi(r) + \sigma_0(r) - 1$ cannot exceed r. There are only r digits for all bases $r \geq 2$. Theorem 2.3 proves that digit 1 is uniquely both a divisor and totative of r. The unit is the only digit counted by both $\phi(r)$ and $\sigma_0(r)$; otherwise the two counting functions are mutually exclusive. Thus the neutral digit function $\nu(r)$ will reveal the quantity of digits which are neither divisors of nor coprime to r. ▲

It is easy to produce large numbers of neutral digits when r is a large composite number, however, $\nu(4) = 0$. The number 4 has 3 divisor digits $\{0, 1, 2\}$ and two totatives $\{1, 3\}$. Theorem 2.3 proves only the digit 1 should appear in both sets. Thus, there is no room in base 4 for neutral digits. We will illustrate why this is so below. The number 6 admits a single neutral digit 4. The divisor digits of base 6 are $\{0, 1, 2, 3\}$ and the totatives are $\{1, 5\}$. Once $r \geq 6$, $\nu(r)$ generally appears to increase, mostly at the expense of $\sigma_0(r)$.

This section has shown that there exist neutral digits in addition to units, divisors, and totatives among digits $0 < n \leq r$ for at least some composite bases r. Observations indicate that the composite number 4 does not possess neutral digits. Larger composite bases r appear to have neutral digits in amounts that increase as r increases.

Figure 2.2. "Countable" digits ϰ of base r = 10, shown in color.

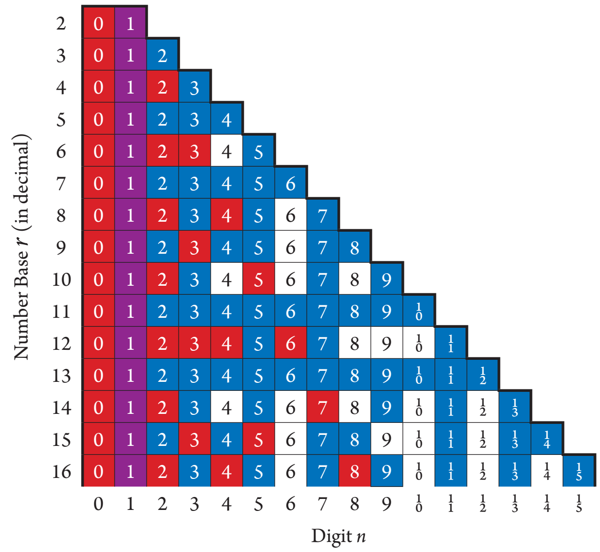


Figure 2.3. "Countable" digits ϰ of bases 2 ≤ r ≤ 16, shown in color. Number bases r are arranged along the vertical axis, increasing from top to bottom. The digits {0, 1, … , ω} of each base r appear along the horizontal axis. Digits d ∣ r are shown in red, while digits t ⊥ r appear in blue. The digit 1 ∣ r ∧ 1 ⊥ r in all bases, and is colored purple in the chart. The digits s of bases r that remain white are neutral digits.

Figure 2.4. Neutral digits s of base r = 10, shown in gold.

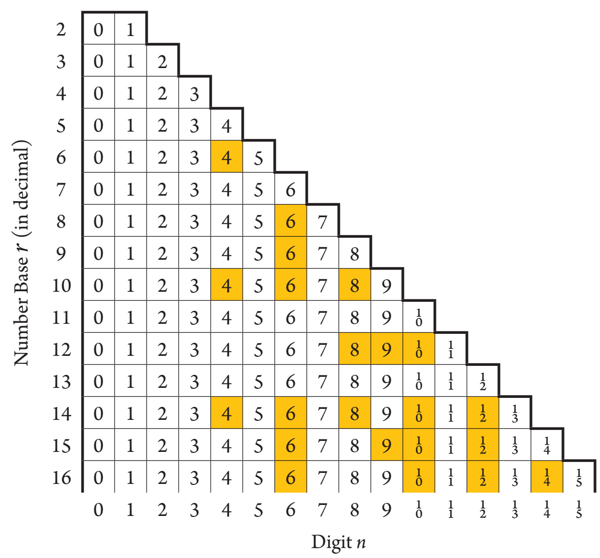


Figure 2.5. Neutral digits s of bases 2 ≤ r ≤ 16, shown in gold.

4. Two Kinds of Neutral Digits.

Recall that digits s that are neither divisors of nor coprime to r must be composite.

Theorem 4.1. There are only two kinds of neutral digit.

Let p be an arbitrary prime number, let $d \mid r$ and $t \perp r$. The minimum number of primes p necessary to furnish a composite product is two. Since a prime $p \mid r \vee p \perp r$ we are restricted to the following three cases:

$$dd, \qquad dt, \qquad tt.$$

(An alternate way to view this is to consider d representing a component which is a homogenous product of prime divisors, and t representing a component that is a homogenous product of prime totatives.)

Consider the case "tt", involving a homogenous product of coprime factors

Lemma 4.1.1. A composite digit n that is the homogenous product of primes $q \perp r$ is a composite totative t_c if and only if $1 \le n \le r$. Such a digit t_c cannot be a neutral digit since $t_c \perp r$.

Proof. It is possible that the prime totatives t may be identical or distinct. Consider the composite homogenous product $1 < n < r$. Let the arbitrary positive integer $k \ge 2$ and the integer $1 \le i \le k$. Then we have

$$t_c = \prod_{i=1}^{k} q_i = q_1 \cdot \ldots \cdot q_k. \tag{4.1}$$

Since each $q_i \perp r$, their product $t_c \perp r$. [31] Clearly, the arrangement "tt" simply describes a composite totative t_c where $r \perp t_c$ since r is coprime to each of the prime divisors q_i of t_c, and all of these prime divisors of t_c are prime totatives q of r.

The definition of a neutral digit is an integer $1 \le n \le r$ that is neither a divisor of nor coprime to r. Since $t_c \perp r$, t_c cannot be a neutral digit. The composite totative t_c is counted by the Euler totient function $\phi(r)$. ▲

We examine the case "dt", involving a mixed product of divisors and coprime factors.

Lemma 4.1.2. A composite number n that is the mixed product of primes $p \mid r$ and $q \perp r$ is a neutral digit if and only if $1 \le n \le r$.

We will call this type of neutral digit a "semitotative," s_t.

Proof. This is equivalent to the case "dt" above. This case regards a composite $1 < pq < r$ produced by some prime divisor p and prime totative q of r. Since $p \mid r$ and $p \mid pq$, p is a common divisor of r and pq. [32] The product pq cannot be coprime to r, since $\gcd(pq, r) = p$, and p by definition is prime so $p \ne 1$. Since $q \mid pq$ but $q \perp r$, $pq \nmid r$. Clearly, pq is a neutral number.

Suppose $pq \mid r$. Let the integer $k \ge 2$ and $1 \le i \le k$. Let the prime $p_i \mid r$. A product pq that would divide r would have the form $pq = p_1^{\delta_1} p_2^{\delta_2} \ldots p_k^{\delta_k}$ per formula (1.3). Since p is prime, it must then be equal to one of the prime divisors p_i. By definition, the prime $q \nmid r \wedge q \ne p_i$, so pq cannot be represented as a product of prime divisors p_i. [33] The representation would need to be $pq = p_1^{\delta_1} p_2^{\delta_2} \ldots p_k^{\delta_k} q$. Thus, $pq \nmid r$.

Since pq is a product of primes, $pq \ne 1$.

Additional prime divisors and prime totatives do not affect the relationship of pq with r. Thus the arrangement pq = "dt" is a composite number that neither divides r evenly, nor is coprime to r.

The product $1 \le pq \le r$ is a digit by definition. A product $pq > r$ is not considered a digit of r, thus cannot be a neutral digit of r. Thus products $1 \le pq \le r$ are a type of neutral digit s_t with a mixed composition, composed of at least one prime divisor p and at least one prime totative q of r. ▲

Such products $pq > r$ are composite with $pq \nmid r$ and pq not coprime to r for the same reasons that apply to $1 < pq < r$. Such a number $pq > r$ can be called a "semi-coprime number".

Let's examine the case "dd", a homogenous product of prime divisors of r.

Definition 4.1.3. This is equivalent to the case "dd" above. Let i and k be integers $(1 \le i \le k)$. Let the positive primes $p_i \mid r$. Consider a positive composite integer g that is a homogenous product of at least 2 arbitrary prime divisors p_i of r. Let the integer $1 \le \delta_i$ be the multiplicity of each p_i in g. Each g thus has a standard form prime decomposition

$$g = \prod_{i=1}^{k} p_i^{\delta_k} = p_1^{\delta_1} p_2^{\delta_2} \ldots p_k^{\delta_k} \quad (p_1 < p_2 < \ldots < p_k) \tag{4.2}$$

Each $p_i \mid g \wedge p_i \mid r$. Let P be the set of distinct prime divisors p_i of r. Then in g, each $p_i \in P$. This number g is called a regular number. [34, 35] A regular number g is a regular digit in base r if and only if $1 \le g \le r$.

Case "dd" thus describes regular digits g of base r. There are two variations of this case, predicated on the relationship of at least one multiplicity δ_i and the corresponding a_i. Two situations are possible, $1 \le \delta_i \le a_i$ and $dd \mid r$ or $\delta_i > a_i$ and $dd \nmid r$.

The first variation concerns g with $1 \le \delta_i \le a_i$. Then the regular digit $g \mid r$, and is thus a composite divisor d_c of r. Let the integer $1 \le \delta_i \le a_i$ be the multiplicity of each of the base r's distinct prime divisors p_i in the divisor d. Each divisor d thus has a standard form prime decomposition shown in formula (1.3):

$$d = p_1^{\delta_1} p_2^{\delta_2} \ldots p_k^{\delta_k}$$
$$(p_1 < p_2 < \ldots < p_k)$$

Lemma 4.1.4. Let the integer $1 \le g$ be a regular number of r. Regular numbers $g \mid r$ if and only if all $\delta_i \le a_i$.

Proof. Each $p_i \mid r \wedge p_i \mid d$, thus there must be an integer $m_i \ge 1$ such that $\gcd(d, r) = m_i p_i$. Formula (1.2) can be rewritten

$$r = d \cdot d' = p_i \cdot m_i \tag{4.3}$$

for each p_i, thus each $m_i \mid r$. Thus far, a number so constructed must be regular, as the formula applying to a divisor (1.3) and the formula applying to a regular number (4.2) are in the same form. Formula (1.3) restricts the multiplicities $1 \le \delta_i \le a_i$. The divisor produced by formula (1.3) can also be produced by formula (4.2). The divisor d is thus a special case of the regular digit g. No number $g > r$ can divide r evenly, since no integer g' can be produced such that $r = g \cdot g'$. Thus a regular number $g \mid r$ implies $g < r$, thereby g is a digit of r. The regular digit $g \mid r$ if and only if all $1 \le \delta_i \le a_i$. ▲

The second variation involves g with $\delta_i > a_i$. One of the distinct prime divisors p_i is "richer" in g than it is in base r. If this proves to be a "rich" neutral digit, we will call it a "semidivisor" s_d.

Lemma 4.1.5. Let i and k be integers $(1 \le i \le k)$. Let the positive primes $p_i \mid r$. A homogenous product g of at least two primes $p_i \mid r$ is a neutral digit if and only if both $\delta_i > a_i$ and $1 \le n \le r$.

Proof. The composite number g must be regular since it can be written in the form presented by formula 4.2:

$$g = \prod_{i=1}^{k} p_i^{\delta_k} = p_1^{\delta_1} p_2^{\delta_2} \dots p_k^{\delta_k}$$
$$(p_1 < p_2 < \dots < p_k)$$

Suppose at least one exponent $\delta_i > a_i$ in the prime decomposition of r. Then the ratio r/g cannot be an integer, which implies $g \nmid r$. Suppose $\delta_1 = a_1 + 1$, and the remaining $\delta_k = a_k$. We can then rewrite r/g as $p_1^{a_1} p_2^{a_2} \dots p_k^{a_k} / p_1^{\delta_1} p_2^{\delta_2} \dots p_k^{\delta_k}$, which simplifies to $p_1^{a_1}/p_1^{(a_1+1)}$. Our inability to obtain an integer from r/g with at least one $\delta_i > a_i$ implies $g \nmid r$. Since no part of g is coprime to r, g is not a semitotative. Clearly g is not itself coprime to r, since $\gcd(g, r) \ne 1$. The regular digit $g \ne 1$, since g is the product of prime numbers. Thus the composite $g = p_1^{\delta_1} p_2^{\delta_2} \dots p_k^{\delta_k}$ with at least one $\delta_i > a_i$ must be a second kind of regular, neutral digit. ▲

Proof of Theorem 4.1. Three possible cases of composite digits *dd*, *dt*, and *tt* have been studied in Lemmas 4.1.1-4.1.5. Lemma 4.1.1 shows that case $tt \perp r$, thus does not represent a neutral digit. Lemma 4.1.2 shows that case *dt* describes a neutral digit involving at least one prime divisor and one prime totative. Definition 4.1.3 is equivalent to case *dd*, and is broken down into two possible situations. Lemma 4.1.4 describes one situation where case *dd* describes regular digits $g \mid r$ that are not neutral by definition. Lemma 4.1.5 describes another situation where case *dd* describes a neutral digit that is a homogenous product $g \nmid r$, the product of at least two primes $p_i \mid r$. No other cases nor situations of cases are possible, hence there are only two kinds of neutral digit, one described by Lemma 4.1.2 and the other described by Lemma 4.1.4. ▲

Section 4 has demonstrated that there are precisely two types of neutral digit. One kind of neutral digit is regular, as it is composed only of elements of the set of distinct prime divisors of r. This is the semidivisor. The other kind of neutral digit is the semitotative, which is the product of at least one prime divisor and at least one totative of r. No other type of neutral digit is possible.

Figure 3.1 shows the decimal semidivisors {4, 8} and the decimal semitotative {6}. Figure 3.2 shows the two types of neutral digits for bases between 2 and 16 inclusive.

Given the possibility of only two kinds of neutral digits, we can draw a "digit map" for each number base r that shows all types of digit relationships with r. We will color-code the digits using the convention in Figure 3. Thus the seven combinations of composition and relationship, the unit, the prime and composite divisor, the semidivisor and the semitotative, and the prime and composite totative, are indicated by such a digit map.

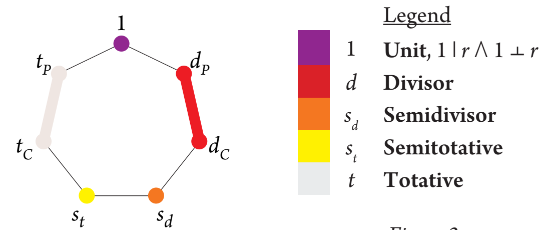


Figure 3.

Figure 3.1. Semidivisors s_d (orange) and semitotatives s_t (yellow) of base r = 10, shown in gold.

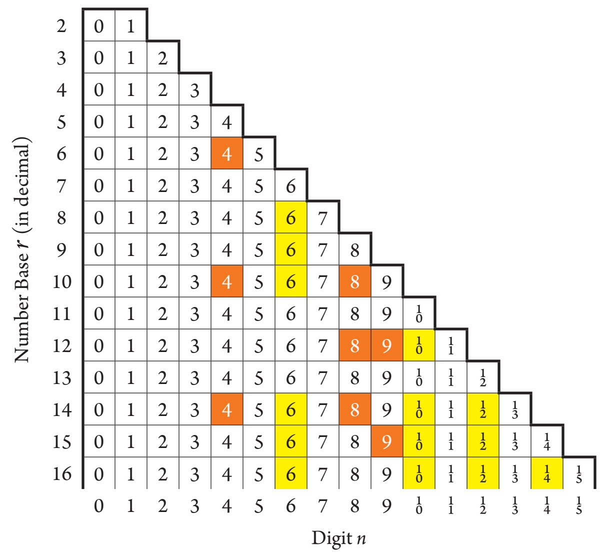


Figure 3.2. Semidivisors s_d (orange) and semitotatives s_t (yellow) of bases $2 \le r \le 16$, shown in color. The digits {0, 1, … , ω} appear in rows, each row a number base r.

Figure 3.3. A digit map of base r = 10.

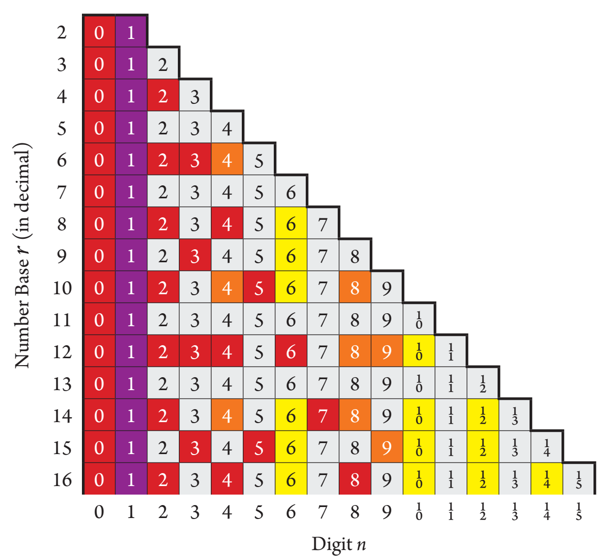


Figure 3.4. A digit map of bases $2 \le r \le 16$, colored according to Figure 3.

Figure 3.3 is a decimal digit map. Figure 3.4 shows digit maps for bases between 2 and 16 inclusive. Table A6 in the Appendix shows digit maps for bases between 2 and 60 inclusive. A bar graph can be generated showing the content of each type of digit in a given base, perhaps called a "digit spectrum" for base r. Table A5 in the Appendix shows digit spectra for $2 \le r \le 60$.

5. The Existence of Semidivisors and Semitotatives.

We will now attempt to prove the existence of semidivisors and semitotatives among composite number bases, recognizing base 4 and the prime bases will prohibit any neutral digit. Before the theorems, let's gather some necessary tools.

Suppose r is a positive composite integer that is the product of k distinct prime divisors, $\{p_1, p_2, \dots p_k\}$, with $p_1 < p_2 < \dots < p_k$. Thus p_1 is the smallest distinct prime divisor of r. Since a composite r must possess at least one divisor less than $r^{1/2}$, it follows that p_1 must be such a divisor. [36] Formula (1.2) shows that each distinct prime divisor d possesses a complement d' such that $r = d \cdot d'$. Let p_1' be the complement to p_1. Thus we have:

$$1 < p_1 \le r^{1/2} \le p_1' < r. \text{ [37, 38]}$$

Note that p_1' is not necessarily prime, since $p_1' = r/p_1$. Given a number base r with at least 3 prime divisors, p_1' is composite. The minimum prime divisor complement p_1' is prime if and only if the number of prime divisors of r is two. These relationships will serve as the basis for several proofs that establish the existence of neutral digits (semidivisors and semitotatives) for certain positive composite integers r.

The Existence of Semidivisors.

A semidivisor is a composite digit s_d which possesses at least one distinct prime factor p having a multiplicity δ which exceeds the multiplicity α of p in the prime decomposition of r. Since prime bases r do not possess neutral digits, only composite bases r may have semidivisors. All we need to do is prove a digit exists which is a power δ of the smallest distinct prime divisor p_1 of r that exceeds the multiplicity α of that same divisor in the prime decomposition of r. Let's test some composite bases to prove the existence of semidivisors.

Theorem 5.1. Let the integer $k \ge 2$ and $1 \le i \le k$. Suppose $r = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \dots \cdot p_k^{\alpha_k}$, $k \ge 2$ and any p_i signifies a distinct prime divisor of r. Such an r having at least two distinct prime divisors will possess at least one semidivisor s_d.

Let's examine the simplest example of a composite number. Suppose r is a positive composite integer that is the product of two prime divisors, p_1 and p_2, where $p_1 \le p_2$. Since a composite r must possess at least one divisor less than or equal to $r^{1/2}$, it follows that p_1 must be such a divisor. Since there are only two nontrivial divisors, we can rewrite formula 1.2 ($r = d \cdot d'$) as $r = p_1 \cdot p_2$. Thus $1 < p_1 \le r^{1/2} \le p_2 < r$.

Lemma 5.1.1. Let p_i signify a prime divisor of r, and let $1 \le i \le 2$. All composite bases $r = p_1 \cdot p_2$ will possess at least one semidivisor s_d.

Proof. Consider the case where $p_1 \ne p_2$. Thus the distinct prime divisors of r are $\{p_1, p_2\}$ and their multiplicities are both 1. It follows that neither p_1 nor p_2 can equal $r^{1/2}$ (if either did, then both would equal $r^{1/2}$ and not be distinct prime divisors.) Thus $1 < p_1 < r^{1/2} < p_2 < r$. Since $r^{1/2} > p_1$, $(r^{1/2})^2 = r$ and $r > p_1^2$. So the square of p_1 must be a digit in base r. Since $p_1 \mid p_1^2 \wedge p_1 \mid r$, p_1^2 is a regular digit of base r. The prime decompositions of $\{p_1^2, r\}$ are $\{p_1^2, p_1 \cdot p_2\}$. Since the multiplicity $\delta_1 > \alpha_1$, p_1^2 is a regular digit that cannot be a divisor of r. We are left to conclude that p_1^2 is a semidivisor of r. Thus, all composite bases r which are products of two prime divisors will possess at least one semidivisor. ▲

Lemma 5.1.2. Let p be a prime divisor of r. Composite bases $r = p^2$ cannot possess semidivisors.

Proof. Let's consider the case where $p_1 = p_2$, thus are the same distinct prime divisor p. Then both p_1 and p_2 must equal $r^{1/2}$ and we have $r = p^2$. Clearly r has only one distinct prime divisor p which when squared equals r itself, and $r \mid r$; both p and p^2 are divisors of r. Therefore a number base r which is the square of a prime divisor p cannot possess semidivisors. ▲

Lemma 5.1.3. Let p be a prime divisor of base r and α the multiplicity of that prime divisor. Suppose $r = p^{\alpha}$. Such an r which is a power of a single prime divisor cannot possess semidivisors.

Proof. The set of divisors of $r = p^{\alpha}$ will be the set of the powers of p^i, $0 \le i \le \alpha$, that is $\{1, p, p^2, p^3, \dots p^{(\alpha-1)}, p^{\alpha}\}$. Since r possesses only one distinct prime divisor, r is restricted only to the powers of p to produce regular digits, and all of these p^i, $0 \le i \le x$ are divisors of r. The multiplicity i of p is less than or equal to the multiplicity α of p in the integer r. Therefore, a base r which is a power of a prime cannot possess semidivisors. ▲

Lemma 5.1.4. Let the number k of distinct prime divisors of r be an arbitrary integer greater than 1 and let the integer $1 \le i \le k$. All composite bases $r = p_1 p_2 \dots p_k$ will possess at least one semidivisor s_d.

Proof. Consider the related case where $r = p_1 p_2 \dots p_k$ and all the p's are distinct prime divisors. If one of the p's, say p_k, is larger than $r^{1/2}$, then the product w of the remaining p's must be less than $r^{1/2}$. It follows that each of the primes $p_1 \dots p_{k-1}$ are smaller than their product w, which itself is less than $r^{1/2}$. Therefore any of the primes $p_1 \dots p_{n-1}$ will yield semidivisors for r when they are squared, and any composite base r which is a product of at least two distinct prime divisors will possess at least one semidivisor. ▲

Proof of Theorem 5.1. Suppose r is a composite number with m prime divisors (including repeated prime divisors, thus $r = 12 = \{2^2 \cdot 3\}$ would present $m = 3$, while $r = 16 = \{2^4\}$ would yield $m = 4$). Suppose also that there exists a minimum prime divisor p_1 such that $p_1 \le r^{(1/m)}$.

Consider the case where r possesses one distinct prime divisor p. This implies p_1 must be p, since there are no other prime divisors which divide r. Suppose there exists a semidivisor s_d where the multiplicity $\delta > a$. Thus we must have at least $\delta = a + 1$, since we require any power of p to remain an integer. The prime decomposition of $r = p^a$ and $s_d = p^\delta = p^{(a+1)}$. Clearly, $p^a < p^{(a+1)}$, thus $r < s_d$, contradicting the definition of a semidivisor s_d as being a kind of digit $0 < n \leq r$.

Thus in the case where r possesses a single prime divisor, there can be no semidivisor. Lemmas 5.1.2 and 5.1.3 support this conclusion.

Suppose r possesses at least two distinct prime divisors $p_1, p_2, \ldots, p_k$ with $p_1 < p_2 < \ldots < p_k$. Consider that the prime decomposition of r is written in standard form per formula 1.1. Suppose $\delta_1 = a_1 + 1$ for a power of the smallest prime divisor p_1^δ. Consider the following cases. In the case of $r = p_1 \cdot p_2$ (the number of distinct prime divisors $k = 2$), $p_1^{(a_1+1)} < p_1 \cdot p_2$ shown by Lemma 5.1.1. In the case of r with $k > 2$, the number $p_1^{(a_1+1)} \cdot p_2 < r$ shown by Lemma 5.1.4. Thus there exists a semidivisor s_d for any $r = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_k^{a_k}$, where $k \geq 2$. ▲

These proofs imply that base 6 is the smallest number base that will possess a semidivisor. Digit 4 base 6 is a semidivisor, since the distinct prime divisor of 4 is 2, which has a multiplicity 2, exceeding that of the distinct prime divisor 2 in 6. Bases 2, 3, and 5 are prime and cannot possess neutral digits, thus they cannot possess semidivisors. Base 4 is a power of a prime and cannot possess semidivisors since the minimum nonunitary power of its smallest prime divisor (2^2) is a divisor of 4. All other squarefree composite bases must possess at least one semidivisor.

The Existence of Semitotatives.

A semitotative is a composite digit s_t which is the product of at least one power of a prime divisor p and at least one power of a prime totative q of r. Let p_1 be the smallest prime divisor of r, and let q_1 be the smallest prime totative of r. There must be a complement p_1' to the minimum prime divisor p_1 such that $r = p_1 \cdot p_1'$. The following formula produces the smallest semi-coprime number h_1 in base r:

(5.2) $$h_1 = p_1 \cdot q_1.$$

If $h_1 <$ r, then h_1 is a digit, and thus a semitotative s_1, since a semitotative is a semi-coprime number $h < r$.

Since prime bases r do not possess neutral digits, only composite bases r may possess semitotatives. The case where $r = 2$ cannot produce semitotatives since 2 does not possess prime totatives. The following proofs pertain to the existence of semitotatives for composite bases r.

Theorem 5.2. A composite number base r possesses at least one semitotative s_t if and only if $q_1 < p_1'$.

Suppose a composite base r possesses a minimum prime divisor $1 < p_1 \leq r^{1/2}$, and a minimum prime totative $1 < q_1 < r$. The minimum prime divisor p_1 cannot exceed the square root of r, since $r = p_1 \cdot d'$. If $p_1 > r^{1/2}$, then r/p_1 would yield a $d' < p_1$, contradicting the definition of p_1. A minimum prime totative $q_1 \neq 1$, since q_1 is prime by definition. Suppose $r^{1/2}$ is an integer; a minimum prime totative $q_1 \neq r^{1/2}$ since $r^{1/2} \mid r$ and $q_1 \perp r$ by definition. A minimum prime totative $q_1 \neq r$ again since by definition $q_1 \perp r$. The divisor pair which includes the minimum prime divisor p_1 is arranged thus:

(5.3) $$1 < p_1 \leq r^{1/2} \leq p_1' < r, \text{ with } 1 < q_1 < r$$

It follows that there are three possible configurations of q_1, p_1, and p_1', since q_1 cannot equal either of p_1 and p_1', otherwise $q_1 \mid r$ and thus contradict the definition of q_1:

$$1 < q_1 < p_1 \qquad p_1 < q_1 < p_1' \qquad p_1' < q_1 < r.$$

Let's examine proofs for each of these cases.

Lemma 5.2.1. Bases r which have a smallest prime totative q_1 where $1 < q_1 < p_1$ have at least one semitotative s_t.

Proof. Any product $s_1 < r$, since $r = p_1 \cdot p_1'$ and q_1 is less than p_1', making s_1 is a digit of base r. Thus any r possessing a minimum prime totative $q_1 < p_1$ will have at least one semitotative s_1. In the case of an odd composite r, the minimum prime totative $q_1 = 2$ will be less than any minimum prime divisor p_1, yielding a semitotative s_1. Note that composite values of r which are even cannot have $q_1 < p_1$, since 2 is the smallest prime. ▲

Lemma 5.2.2. Bases r which have a smallest prime totative $p_1 < q_1 < p_1'$ have at least one semitotative s_t.

Proof. This case has the minimum prime totative q_1 interposing between the minimum prime divisor p_1 and its complement p_1'. Note that if $p_1 = r^{1/2}$, then $p_1 = p_1'$ and $p_1 < q_1 < p_1'$ is impossible. Again, any product $s_1 = p_1 \cdot q_1$ would be less than r, since $r = p_1 \cdot p_1' \wedge q_1 < p_1'$, thus $s_1 < r$, making s_1 a digit of base r. Thus any r possessing a minimum prime totative q_1 such that it is greater than its minumum prime divisor p_1 but less than the complement to its minimum prime divisor p_1' will have at least one semitotative s_{min}. ▲

Lemma 5.2.3. Bases r which have a smallest prime totative $p_1' < q_1 < r$ cannot possess any semitotative s_t.

Proof. Since $q_1 > p_1'$ and given $r = p_1 \cdot p_1'$, any product $s_1 = p_1 \cdot q_1$ will exceed r and not be a digit of r. Thus any number base r having its smallest prime totative greater than both the minimum prime divisor and its complement will not have semitotatives. ▲

Proof of Theorem 5.2. Let the integers $k \geq 1$ and $1 \leq i \leq k$. Let p_1 signify the smallest prime (2), p_2 the second smallest (3), etc., and p_k the maximum prime divisor of r_k. Suppose r_k is a primorial, that is, the product of distinct prime factors $p_1, p_2, \ldots p_k$. Such a number r_k will then have $p_1 = 2$ and $p_1' = r_k / p_1 = p_2, p_3, \ldots p_k$.

For $k = 1$, $r_1 = 2$ is itself prime, thus cannot feature neutral digits nor semitotatives, which are a kind of neutral digit.

At $k = 2$ ($r_2 = 6$), and $p_1 < p_1' < q_1$ we have the situation described in Lemma 5.2.3. Because 6 is the product of two primes, p_1' must be 3 and there are no primes between 2 and 3, thus 6 cannot possess semitotatives since $q_1 = 5$ is larger than $p_1' = 3$. (See Figure 5.2 on page 10).

At $k = 3$ ($r_2 = 30$), there exists a significant difference between $p_i = 5$ and $p_1' = 15$, allowing the smallest unrepresented

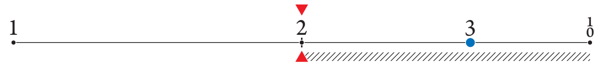


Figure 5.1. Base 4 cannot have semitotatives because the minimum totative q₁ (indicated by ●3) is larger than both the minimum divisor p₁ (▼2) and its complement p′₁ (▲2).

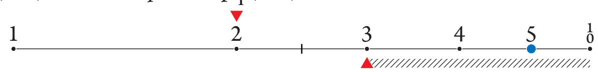


Figure 5.2. Base 6 cannot have semitotatives because q_1 (●5) is larger than both the minimum divisor p_1 (▼2) and its complement p'_1 (▲3).

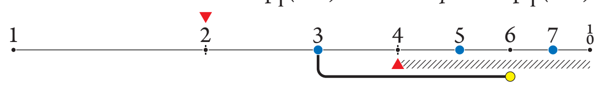


Figure 5.3. Base 8 possesses the semitotative h_1 = ●6 because q_1 (●3) is less than the minimum divisor's complement p'_1 (▲4).

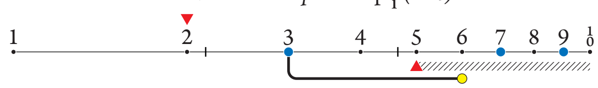


Figure 5.4. Base 10 possesses the semitotative h_1 = ●6 because q_1 (●3) is less than the minimum divisor's complement p'_1 (▲5).

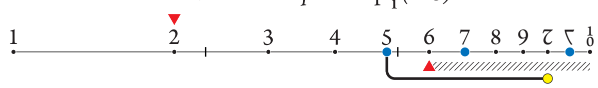


Figure 5.5. Base 12 possesses the semitotative h_1 = ●10 because q_1 (●5) is less than the minimum divisor's complement p'_1 (▲6).

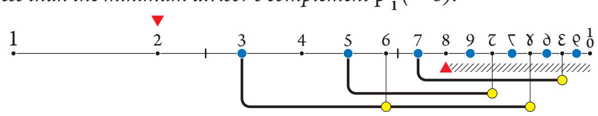


Figure 5.6. Base 16 has four semitotatives. The totatives ●3, ●5, and ●7 are less than p'_1 = ▲8. The totative ●3 generates the semitotatives ●6 and ●12 by multiplication with the divisors ●2 and ●2² respectively. The totatives ●5 and ●7 produce the semitotatives ●10 and ●14 respectively with the divisor ●2.

prime $q_1 = p_{k+1} = 7$ to produce a smallest semidivisor $s_1 = 14$. Thus at $i = 3$ we have the condition $p_i < q_1 < p'_{1'}$ presented in Lemma 5.2.2. As the value of i increases, the difference $\delta_s = p'_1 - p_k$ widens, since only a few of the smallest primes will produce a value of r_k large enough to admit a p_{k+1} and therefore a minimum semitotative s_1. Thus any primorial $r_k > 6$ will have at least one semitotative.

Cases where the multiplicity α_i of any of the distinct prime divisors p_i of r_k are increased, e.g., $\{2^2 \cdot 3 \cdot 5\} = 60$ versus $r_3 = \{2 \cdot 3 \cdot 5\} = 30$ only increase the difference $p'_1 - p_1$. Cases where some p_i, where $p_1 < p_i < p_k$ is left out, e.g., $\{2 \cdot 5\} = 10$ versus $r_3 = \{2 \cdot 3 \cdot 5\} = 30$, admit $q_1 < p'_1$.

The case where r is odd, meaning the smallest prime totative $q_1 = 2$ yields the condition described in Lemma 5.2.1. Leaving ensuing small primes q_i out of the prime decomposition of r only furnishes more small totatives that produce semitotatives with the minimum prime divisor p_1. Therefore all composite numbers greater than 6 will possess at least one semitotative. ▲

Corollary 5.3. Bases 4 and 6 are the only composite values of r that do not possess semitotatives. Semitotatives do not exist for bases $2 \le r < 8$.

Proof of Case $r = 4$. When $r = 4$, the minimum prime divisor $p_1 = 2 = r^{1/2}$, thus $p_1 = p'_{1'}$ while the minimum prime totative $q_1 > p'_1$ (i.e. 3 > 2). The product $s_1 = p_1 \cdot q_1 = 2 \cdot 3$ exceeds 4. Thus 4 cannot possess a semitotative, though it is a composite r. Since $r = 4$ cannot possess a semidivisor as described above, such an r is an instance of a composite r that cannot possess neutral digits. (See Figure 5.1). ▲

Proof of Case $r = 6$. When $r = 6$, the minimum prime divisor $p_1 = 2$. The complement $p'_1 = 3$, while the minimum prime totative $q_1 = 5$ is larger than both. The product $s_1 = p_1 \cdot q_1 = 2 \cdot 5$ exceeds 6. Thus 6 cannot possess a semitotative, though it does possess a semidivisor (i.e., the digit 4).

No prime base $r = p$ can possess semitotatives because neutral digits do not exist in prime number bases. Additionally, the only prime divisor available is p. Section 2 and the The Euler totient function $\phi(p)$ for a prime p shown in formula 2.5 show that for values $0 < n < r, n \perp r$. All primes p have two positive divisors $\{1, p\}$. There is no room for semitotatives in prime bases p. Thus, semitotatives do not exist for all bases $2 \le r < 8$. ▲

Theorem 5.4. At least 1 neutral digit s exists for all composite number bases $r > 6$.

Proof. Theorem 5.1 shows that there exist at least one semidivisor s_d for all squarefree composite integers $r > 2$. Theorem 5.2 proves that there exists at least one semitotative s_t for some positive composite values of r if and only if the minimum prime totative $q_1 < p'_{1'}$ the complement of the minimum prime divisor of r. Corollary 5.3 shows that semitotatives do not exist for all bases $2 \le r < 8$. Thus there exist at least one neutral digit s for all composite number bases $r > 6$. (See Figure 5.2).▲

This section has demonstrated that all squarefree composite bases r must possess at least one semidivisor, and that all composite numbers $r > 6$ possess at least one semitotative. This aligns with the assertion in the previous section that all composite numbers except 4 possess neutral digits. The number base 6 is unique in that it possesses a semidivisor (the senary digit 4) but no semitotative. Bases which are powers of a single prime divisor (except base 4) possess semitotatives but no semidivisors. Base 8 is the smallest instance of a number base possessing a semitotative—the octal digit 6. (See Figure 5.3).

6. Construction and Quantification of Neutral Digits

This section will illustrate the construction of the set of all semidivisors and that of all semitotatives of base r.

Construction of the Set of Semidivisors

Here is a recapitulation of a few facts about a semidivisor s_d and the number base r. The semidivisor s_d is a digit, thus $0 < s_d < r$. Lemma 4.1.3 shows that s_d does not divide r evenly, nor is s_d coprime to r, thus it is a kind of neutral digit. (The concept of a neutral digit was established in Section 2 and can be counted using Formula 3.1 of Theorem 3.1.) The standard form of prime decomposition of any positive integer $r \ge 2$ is given by Formula 1.1. Let p be a distinct prime divisor of r.

Let the integer k be the number of distinct prime divisors p of r. Let the integer $0 < i \leq k$. Let α_i be the exponent of the prime divisor p_i. Then the number base r has the standard form of prime decomposition

$$r = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$$
$$(\alpha_1 > 0, \alpha_2 > 0, \dots, \alpha_k > 0, p_1 < p_2 < \dots < p_k).$$

Regular numbers are described by Definition 4.1.3 and Formula 4.2. Let δ_i be the exponent of the prime divisor p_i. Thus the regular number g has the prime decomposition

$$g = \prod_{i=1}^{k} p_i^{\delta_k} = p_1^{\delta_1} p_2^{\delta_2} \dots p_k^{\delta_k}$$
$$(p_1 < p_2 < \dots < p_k).$$

The semidivisor is a regular neutral digit s_d that has at least one prime divisor p_i with multiplicity $\delta_i > \alpha_i$ that is the multiplicity of the corresponding p_i in r. Every prime divisor p of s_d is also found in the prime decomposition of r such that $p \mid s_d \land p \mid r$.

We can generate the set S_d of semidivisors of r through the following technique. Firstly, consider a matrix of products of the powers of each distinct prime divisor p_i of r. The integer k is the number of distinct prime divisors p of r. Such a matrix is k-dimensional, requiring an axis for each distinct prime divisor of r. For $r = 10$, two axes are required, since the distinct prime divisors of 10 are $\{2, 5\}$. For $r = 60$, the matrix is three dimensional, since the distinct prime divisors of 60 are $\{2, 3, 5\}$. Presuming there exist two distinct prime divisors a and b of r, the matrix of regular numbers of such an r can be arranged as shown by Figure 6.1. Suppose r has three distinct prime divisors a, b, and c. Then the matrix of regular numbers will be three dimensional, and will resemble the series of two dimensional matrices shown in Figure 6.2. Figure 3 illustrates the regular digits of base $r = 10$.

The regular digit is a regular number $g \leq r$, thus using the example of $r = 10$, the matrix of regular numbers can be truncated such that all products in the table are less than or equal to r. Since we are dealing strictly with digits, the digit zero can substitute for r. This is because digit zero signifies congruence with r. (The special case where digit zero represents actual zero is ignored.) Figure 6.4 is a table of regular digits for $r = 10$.

The divisor of r is a special case of a regular digit, described by Formula 1.3, recapitulated here. Let the integer $1 \leq \delta_i \leq \alpha_i$ be the multiplicity of each of the base r's distinct prime divisors p_i in the divisor d. Each divisor d thus has a standard form prime decomposition

$$d = p_1^{\delta_1} p_2^{\delta_2} \dots p_r^{\delta_r}$$
$$(p_1 < p_2 < \dots < p_k)$$

Thus, the table of divisors is contained in the table of regular digits, since no exponent $\delta_i > \alpha_i$. All products of prime divisors having $1 \leq \delta_i \leq \alpha_i$ thereby occupy an orthogonal k-dimensional region of the matrix of regular digits. For a base r that has two distinct prime factors, the divisors occupy a rectangle; if r is square, the divisors occupy a square. A square r that has three distinct prime factors has a cubic matrix of divisors. Figure 6.5

	a^0	a^1	a^2	a^3	a^4	…
b^0	1	a	a^2	a^3	a^4	…
b^1	b	ab	a^2b	a^3b	a^4b	…
b^2	b^2	ab^2	a^2b^2	a^3b^2	a^4b^2	…
b^3	b^3	ab^3	a^2b^3	a^3b^3	a^4b^3	…
b^4	b^4	ab^4	a^2b^4	a^3b^4	a^4b^4	…
⋮	⋮	⋮	⋮	⋮	⋮	

Figure 6.1. An infinite two dimensional matrix of regular numbers in base r *with two distinct prime divisors* a *and* b.

$(\times c^0)$	a^0	a^1	a^2	…
b^0	1	a	a^2	…
b^1	b	ab	a^2b	…
b^2	b^2	ab^2	a^2b^2	…
⋮	⋮	⋮	⋮	

$(\times c^1)$	a^0	a^1	a^2	…
b^0	c	ac	a^2c	…
b^1	bc	abc	a^2bc	…
b^2	b^2c	ab^2c	a^2b^2c	…
⋮	⋮	⋮	⋮	

$(\times c^2)$	a^0	a^1	a^2	…
b^0	c^2	ac^2	a^2c^2	…
b^1	bc^2	abc^2	a^2bc^2	…
b^2	b^2c^2	ab^2c^2	$a^2b^2c^2$	…
⋮	⋮	⋮	⋮	

Figure 6.2. A three dimensional matrix of regular numbers in base r *with three distinct prime divisors* a, b, *and* c.

	2^0	2^1	2^2	2^3	2^4	2^5	2^6	…
5^0	1	2	4	8	16	32	64	…
5^1	5	10	20	40	80	160	320	…
5^2	25	50	100	200	400	800	1600	…
5^3	125	250	500	1000	2000	4000	8000	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Figure 6.3. A matrix of decimal regular numbers.

	2^0	2^1	2^2	2^3
5^0	1	2	4	8
5^1	5	0		

Figure 6.4. A matrix of decimal regular digits. A regular digit is a regular number g ≤ r. *The number "10" is replaced by digit zero, which signifies congruence with base* r.

	2^0	2^1	2^2	2^3
5^0	**1**	**2**	4	8
5^1	**5**	**0**		

Figure 6.5. A matrix of decimal regular digits, highlighting the region containing the divisors of r*. Digits lying outside this region are semidivisors of base* r*. Thus the set of decimal semidivisors is {4, 8}.*

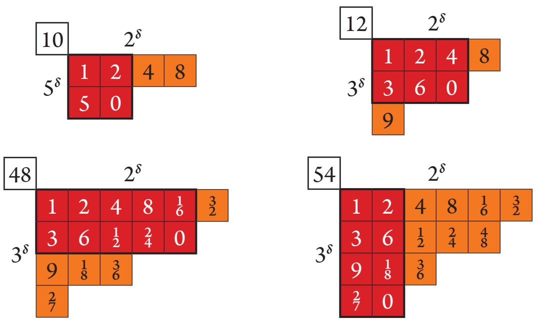


Figure 6.6. Maps of regular digits in bases 10, 12, 48, and 54. Divisors and semidivisors appear on red and orange backgrounds, respectively.

shows the decimal divisors in boldface, with a thick line denoting the region of divisors. The digits outside of the region of divisors are the decimal semidivisors.

Figure 6.6 includes matrices of regular digits for some values of r. The cells of these matrices are color coded such that divisors are printed in white on red backgrounds, and semidivisors are printed in black on orange backgrounds. The decimal value of each digit is shown. We will call digits $n \geq 10$ "transdecimal" digits, as they find application in bases $r > 10$. The transdecimal digits of number bases may employ a set of arbitrary numerals. In Figure 6.6, the numerals are simply "stacked" decimal values, the tens place over the ones place. Such matrices may be called "regular digit maps" for base r. Appendix Table A2 furnishes regular digit maps for $6 < r < 66$. Indeed, regular digit maps can become involved: Appendix Table A3 shows a regular digit map for base 2520.

Construction of the Set of Semitotatives

A semi-coprime number h is a composite number having at least one prime divisor $p \mid r$ and at least one prime divisor $q \perp r$. The semitotative s_t is simply a semi-coprime number $h < r$. Thus a semitotative is a composite neutral digit that has at least one prime divisor $p \mid r$ and at least one prime divisor $q \perp r$.

At least two methods exist for construction of the set of semitotatives S_t of a number base r. We can construct a matrix of products of totatives t and regular digits g which will produce semi-coprime numbers of base r. Multiplication of the prime divisors p_i and the prime totatives q_i of r is also sufficient if multiple instances of p_i and q_i are allowed. We will focus on the first concept.

A two dimensional matrix of semi-coprime numbers h in base r can be computed by arranging the set G of regular numbers g in base r along one axis and the set C of numbers $c \perp r$ on another axis. Let g_1 be the minimum regular number and c_1 be the minimum coprime number in base r. Let the integers $i > 0$ and $j > 0$. Then one axis can have the infinite series $g_1, g_2, \ldots, g_i, \ldots$ while the other has the infinite series $c_1, c_2, \ldots, c_j, \ldots$. Figure 6.7 illustrates such a matrix of semi-coprime numbers.

A matrix of semitotatives is simply a semi-coprime matrix truncated at values that exceed r. Let p_1 be the minimum prime divisor of r, and let q_1 be the smallest prime totative of r. There must be a complement p_1' to the minimum prime divisor p_1 such that $r = p_1 \cdot p_1'$. Let the integer t' be the "totative complement to p_1", the largest totative less than p_1'. Let the integer g' be the largest regular digit less than r/t_1. The matrix of semitotatives will include all regular numbers $p_1 \leq g \leq g'$ on one axis and all totatives $t_1 \leq t \leq t'$ on the other axis. Figure 6.9 shows some semitotative matrices in the form of semitotative maps. Appendix Table A4 shows semitotative maps for $8 \leq r \leq 60$.

A Test for Semidivisors and Semitotatives

Recall the test for a neutral digit in formula 2.11:

$$1 < \gcd(n, r) < n.$$

We can write a recursive routine that would determine whether an integer $n > 0$ is a regular number $g \nmid r$ or a semi-coprime number h using the digit n, the base r, and the gcd function. Let the integer $i \geq 1$.

Step 1: let $n_1 = n/\gcd(n, r)$ if $1 < \gcd(n, r) < n$.

Step 2: Check n_i. If any $n_i = 1$, then n is a regular number $g \nmid r$. If $g < r$, then g is a semidivisor s_d.

Step 3: Check $\gcd(n_i, r)$. If $i \geq 1 \wedge \gcd(n_i, r) = 1$, then the routine has found a semi-coprime number h. If $h < r$, then h is a semitotative s_t.

Step 4: Iterate until conditions in Step 2 or 3 materialize. Let $n_{(i+1)} = n_i/\gcd(n_i, r)$ if $1 < \gcd(n_i, r) < n$. Return to Step 2.

Counting Semidivisors

Suppose there exists a counting function $\rho(r)$ that enumerates the regular digits of r. Let the integer $l > 1$ and let $\text{Floor}(x)$ be a function which takes the integer part of a real number x. The greatest integral power $\delta_{i,j}$ of a prime divisor p_i can be determined using the following:

$$\delta_{i,l} = \text{Floor}\left(\frac{\operatorname{Log} r}{\operatorname{Log} p_i}\right). \tag{6.1}$$

There are thus $\delta_{i,l} + 1$ elements along each axis, defining the maximum range of the matrix for p_i. This paper will not define a regular digit counting function $\rho(r)$. For the current purpose the quantity of regular digits $\rho(r)$ will be tallied manually.

We can derive a counting function $\nu_d(r)$ for the number of semidivisors s_d of r using the following formula:

$$\nu_d(r) = \rho(r) - \sigma_0(r). \tag{6.2}$$

Counting Semitotatives

A counting function akin to that for regular digits can be devised, using the method presented above for constructing all semitotatives of *r*. That method is reliant on knowledge of regular numbers. Theorem 4.1 proves that there are only two kinds of neutral digit. Thus, using the neutral digit function established in formula 3.1,

$$\nu(r) = r - [\phi(r) + \sigma_0(r) - 1],$$

we can compute the number of semitotatives using the following formula:

$$\nu_t(r) = \nu(r) - \nu_d(r). \tag{6.3}$$

This section has demonstrated some methods for constructing the set S_d of semidivisors and S_t of semitotatives of *r*. A routine that tests whether an integer $n > 0$ is a regular number $g \nmid r$ or a semi-coprime number *h* if $1 < \gcd(n, r) < n$. Functions were defined that quantify semidivisors and semitotatives based on the neutral digit function defined by formula 3.1.

7. Conclusion.

This work proves the existence of neutral digits in composite bases except base 4. Section 2 demonstrates that for some composite numbers $r \geq 2$, there exist digits $0 < n \leq r$ that are neither divisors of nor coprime to *r*. In Section 3 we define a neutral digit counting function. We prove in Section 4 there are two and only two types of neutral digits, both composite. One kind of neutral digit is the semitotative s_t whose prime factors include at least one prime divisor *p* and at least one prime totative *q* of *r*. The other kind is a semidivisor s_d, a regular digit $g \nmid r$. Section 5 proves semidivisors exist for all squarefree composite bases *r* except for $r = 4$. We prove semitotatives exist for all composite bases *r* except for $r = 4$ and $r = 6$, and we explore the reasons why bases 4 and 6 are exceptions. Jointly, the proofs in Section 5 prove at least 1 neutral digit *s* exists for all composite number bases $r > 6$. Section 6 illustrates methods of construction of the sets of semidivisors and ssemitotatives in base *r*. Counting functions for each kind of neutral digit are created based on the neutral digit function. The Appendix Table 5 summarizes the kinds of digits for bases $2 \leq r \leq 60$, while Table 6 maps out each digit in the same range of bases.

Work can be done to produce algorithms that analyze the relationships of digits *n* to bases *r*. Further work on an algorithm that produces digit maps and spectra for bases *r* can be pursued. A solid counting function for semidivisors and semitotatives would prove handy. Initially this work may consist of simply counting digits of a certain kind in a digit map or spectrum. These algorithms are currently under development in Mathematica.

This work serves as one part of a foundation for future work regarding number bases. A second paper will cover indirect relationships between digits of base *r* and the numbers $r - 1$ and $r + 1$. It will also cover inherited divisibility rules.

Michael Thomas De Vlieger,
St. Louis, MO, 23 November 2011.

	g_1	g_2	g_3	g_4	g_5	…
c_1	c_1g_1	c_1g_2	c_1g_3	c_1g_4	c_1g_5	…
c_2	c_2g_1	c_2g_2	c_2g_3	c_2g_4	c_2g_5	…
c_3	c_3g_1	c_3g_2	c_3g_3	c_3g_4	c_3g_5	…
c_4	c_4g_1	c_4g_2	c_4g_3	c_4g_4	c_4g_5	…
c_5	c_5g_1	c_5g_2	c_5g_3	c_5g_4	c_5g_5	…
⋮	⋮	⋮	⋮	⋮	⋮	

Figure 6.7. An infinite two dimensional matrix of semi-coprime products h *in base* r *having as multiplicands the regular number* g_i *and the coprime number* c_j.

	2	4	5	8	10	…
3	6	12	15	24	30	…
7	14	28	35	56	70	…
9	18	36	45	72	90	…
11	22	44	55	88	110	…
13	26	52	65	104	130	…
⋮	⋮	⋮	⋮	⋮	⋮	

Figure 6.8. A matrix of decimal semi-coprime numbers.

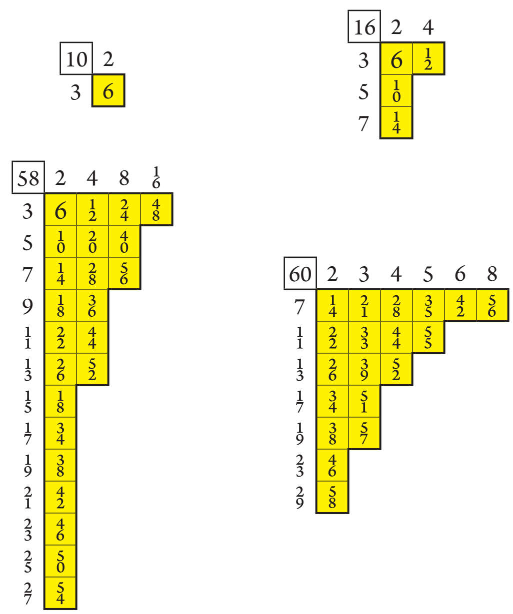


Figure 6.9. Semitotative maps for bases 10, 16, 58, and 60.

1 Weisstein, Eric W. "Digit", Wolfram MathWorld. Retrieved February 2011, < http://mathworld.wolfram.com/Digit.html >.

2 LeVeque 1962, Chapter 1, "Foundation", Section 1–5, "Radix representation", page 19, specifically: "…we can construct a system of names or symbols for the positive integers in the following way. We choose arbitrary symbols to stand for the digits (i.e., the nonnegative integers less than g) and replace the number $c_0 + c_1 g + \ldots + c_n g^n$ by the simpler symbol $c_n c_{n-1} \ldots c_1 c_0$. For example, by choosing g to be ten and giving the smaller integers their customary symbols, we have the ordinary decimal system … But there is no reason why we must use ten as the base or radix…"

3 Dudley 1969, Section 2, "Unique factorization", page 10, specifically: "A prime is an integer that is greater than 1 and has no positive divisors other than 1 and itself".

4 Hardy & Wright 2008, Chapter I, "The Series of Primes", Section 1.2, "Prime numbers", page 2, specifically: "A [positive integer] p is said to be prime if (i) $p > 1$, (ii) p has no positive divisors except 1 and p."

5 Jones & Jones 2005, Chapter 2, "Prime Numbers", Section 2.1, "Prime numbers and prime-power factorisations", pages 19–20, specifically: "Definition: An integer $p > 1$ is said to be prime if the only positive divisors of p are 1 and p itself. … The smallest prime is 2, and all the other primes … are odd."

6 Ore 1948, Chapter 4, "Prime Numbers", page 50, specifically: "An integer $p > 1$ is called a prime number or simply prime when its only divisors are the trivial ones, ± 1 and $\pm p$."

7 Jones & Jones 2005, Chapter 2, "Prime Numbers", Section 2.1, "Prime numbers and prime-power factorisations", page 20, specifically: "An integer $n > 1$ which is not prime … is said to be composite; such an integer has the form $n = ab$ where $1 < a < n$ and $1 < b < n$."

8 Hardy & Wright 2008, Chapter I, "The Series of Primes", Section 1.2, "Prime numbers", page 2, specifically: "[An integer] greater than 1 and not prime is called composite."

9 Dudley 1969, Section 2, "Unique factorization", page 10, specifically: "An integer that is greater than 1 but is not prime is called composite".

10 Weisstein, Eric W. "Composite Number", Wolfram MathWorld. Retrieved February 2011, < http://mathworld.wolfram.com/CompositeNumber.html >.

11 Ore 1948, Chapter 4, "Prime Numbers", page 50, specifically: "A number $m > 1$ that is not a prime is called composite."

12 Dudley 1969, Section 2, "Unique factorization", page 10, specifically: "Note that we call 1 neither prime nor composite. Although it has no positive divisors other than 1 and itself, including it among the primes would make the statement of some theorems inconvenient, in particular, the unique factorization theorem. We will call 1 a unit."

13 Hardy & Wright 2008, Chapter I, "The Series of Primes", Section 1.2, "Prime numbers", page 2, specifically: "It is important to observe that 1 is not reckoned as a prime."

14 Jones & Jones 2005, Chapter 2, "Prime Numbers", Section 2.1, "Prime numbers and prime-power factorisations", page 20, specifically: "Note that 1 is not prime."]

15 Ore 1948, Chapter 4, "Prime Numbers", page 50, specifically: "Lemma 4-1. A prime p is either relatively prime to a number or divides it" and ensuing proof.

16 Hardy & Wright 2008, Chapter I, "The Series of Primes", Section 1.2, "Prime numbers", page 3, specifically: "If we arrange [the primes] in increasing order, associate sets of equal primes into single factors, and change the notation appropriately, we obtain (1.2.2) $n = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$ ($a_1 > 0, a_2 > 0, \ldots, p_1 < p_2 < \ldots$). We then say that n is expressed in standard form", and preceding material on the same page.]

17 Ore 1948, Chapter 5, "The aliquot parts", page 86, specifically, Equation 5-1: "A number N shall be written $N = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$, $\alpha_r \geq 0$, where p_i = the various different prime factors and α_i the *multiplicity*, i.e., the number of times p_i occurs in the prime factorization."

18 Ore 1948, Chapter 5, "The aliquot parts", page 86, specifically, Equation 5-2: "For any divisor d of N one has $N = dd_1$, where d_1 is the divisor paired with d. When multiplied together, the prime factorizations of d and d_1 must give that of N so that $d = p_1^{\delta_1} p_2^{\delta_2} \ldots p_r^{\delta_r}$ where the exponents δ_i do not exceed the corresponding α_i in (5-1). Since the second factor in must contain the remaining factors, it becomes $d_1 = p_1^{(\alpha_1 - \delta_1)} p_2^{(\alpha_2 - \delta_2)} \ldots p_r^{(\alpha_r - \delta_r)}$."

19 Dudley 1969, Section 7, "The divisors of an integer", page 50, specifically: "Let n be a positive integer. Let $d(n)$ denote the number of positive divisors of n (including 1 and n), and let $\sigma(n)$, denote the sum of the positive divisors of n. That is, $d(n) = \sum_{d|n} 1$ and $\sigma(n) = \sum_{d|n} d$. $\sum_{d|n}$ means the sum over the positive divisors of n."

20 Weisstein, Eric W. "Divisor Function", Wolfram MathWorld. Retrieved February 2011, < http://mathworld.wolfram.com/DivisorFunction.html >, specifically for the $\sigma_0(r)$ and $\sigma_1(r)$ notation, equivalent to Dudley's $d(n)$ and $\sigma(n)$, respectively, and Ore's $\nu(n)$ and $\sigma(n)$, respectively.

21 Ore 1948, Chapter 2, "Properties of numbers. Division", page 29, specifically: "Each number has the obvious decomposition $c = 1 \cdot c = (-1)(-c)$ and ± 1 together with $\pm c$ are called trivial divisors."

22 Ore 1948, Chapter 5, "The aliquot parts", page 86, specifically, Equation 5-4: "Theorem 5-1. The number of divisors of a number r in the form [$N = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$] is $\nu(N) = (\alpha_1 + 1)(\alpha_2 + 1) \ldots (\alpha_r + 1)$"]

23 Weisstein, Eric W. "Squarefree", Wolfram MathWorld. Retrieved February 2011, <http://mathworld.wolfram.com/Squarefree.html>.

24 Ore 1948, Chapter 5, "The aliquot parts", pages 109–110, specifically: "5-5. Euler's function. When m is some integer, we shall consider the problem of finding how many of the numbers 1, 2, 3, …, $(m-1)$, m are relatively prime to m. This number is usually denoted by $\varphi(m)$, and it is known as Euler's φ-function of m because Euler around 1760 for the first time proposed the question and gave its solution. Other names, for instance, indicator or totient have been used."

25 Jones & Jones 2005, Chapter 5, "Euler's Function", Section 5.2, "Euler's function", page 85, specifically: "Definition: We define $\phi(n) = |U_n|$, the number of units in $\mathbb{Z}_n$ … this is the number of integers $a = 1, 2, \ldots, n$ such that $\gcd(a, n) = 1$" and the ensuing proof and example.

26 Ore 1948, Chapter 5, "The aliquot parts", page 111, specifically: "When $m = p$ is a prime, all numbers … except the last are relatively prime to p; consequently, $\varphi(p) = p - 1$"

27 Ore 1948, Chapter 5, “The aliquot parts”, page 113, specifically: “Theorem 5-11. Let m be an integer whose various prime factors are $p_1, \ldots, p_r$. Then there are $\varphi(m) = m(1 - 1/p_1) \ldots (1 - 1/p_r)$ integers less than and relatively prime to m.”

28 Dudley 1969, Section 9, “Euler’s Theorem and Function”, page 70, specifically, “If $n = p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}$, then $\phi(n) = n(1 - 1/p_1)(1 - 1/p_2) \ldots (1 - 1/p_k)$.”

29 LeVeque 1962, Chapter 1, “Foundation”, Section 1–3, “Proofs by Induction”, pages 11–12, specifically: “For every positive integer n, u_n and u_{n+1} have no common factor greater than 1” and the ensuing proof.

30 Dudley 1969, Section 2, “Unique factorization”, page 10, specifically: “Note that we call 1 neither prime nor composite. Although it has no positive divisors other than 1 and itself, including it among the primes would make the statement of some theorems inconvenient, in particular, the unique factorization theorem. We will call 1 a unit.”

31 LeVeque 1962, Chapter 2, “The Euclidean Algorithm and Its Consequences”, Section 2–2, “The Euclidean algorithm and greatest common divisor”, pages 24, specifically: “(e) if a given integer is relatively prime to each of several others, it is relatively prime to their product.” and the ensuing example.

32 LeVeque 1962, Chapter 1.1, “The Euclidean Algorithm and Its Consequences”, Section 2–1, “Divisibility”, page 22, specifically: “(3) If $a|b$ and $a|c$, then $a|(bx + cy)$ for each x, y. (If $a|b$ and $a|c$, then a is said to be a *common divisor* of b and c)”

33 Ore 1948, Chapter 5, “The aliquot parts”, page 86, specifically, Equation 5-2: “For any divisor d of N one has $N = dd_1$, where d_1 is the divisor paired with d. When multiplied together, the prime factorizations of d and d_1 must give that of N so that $d = p_1^{\delta_1} p_2^{\delta_2} \ldots p_r^{\delta_r}$ where the exponents δ_i do not exceed the corresponding a_i in (5-1). Since the second factor in $[N = dd_1]$ must contain the remaining factors, it becomes $d_1 = p_1^{(a_1-\delta_1)} p_2^{(a_2-\delta_2)} \ldots p_r^{(a_r-\delta_r)}$”

34 Ore 1948, Chapter 13, “Theory of Decimal Expansions”, page 316, specifically: “In general, let us say that a number is *regular* with respect to some base number b when it can be expanded in the corresponding number system with a finite number of negative powers of b. … one concludes that the regular numbers are the fractions $r = p/q$, where q contains no other prime factors other than those that divide b.”

35 Weisstein, Eric W. “Regular Number”, Wolfram MathWorld. Retrieved February 2011, <http://mathworld.wolfram.com/RegularNumber.html>.

36 Dudley 1969, Section 2, “Unique Factorization”, page 13, specifically: “Lemma 3. If n is composite, then it has a divisor d such that $1 < d_1 \leq n^{1/2}$” and the ensuing proof.

37 Dudley 1969, Section 2, “Unique Factorization”, page 13, specifically: “Lemma 3. If n is composite, then it has a divisor d such that $1 < d_1 \leq n^{1/2}$” and the ensuing proof.

38 Dudley 1969, Section 2, “Unique Factorization”, page 13, specifically: “Lemma 4. If n is composite, then it has a prime divisor less than or equal to $n^{1/2}$” and the ensuing proof.

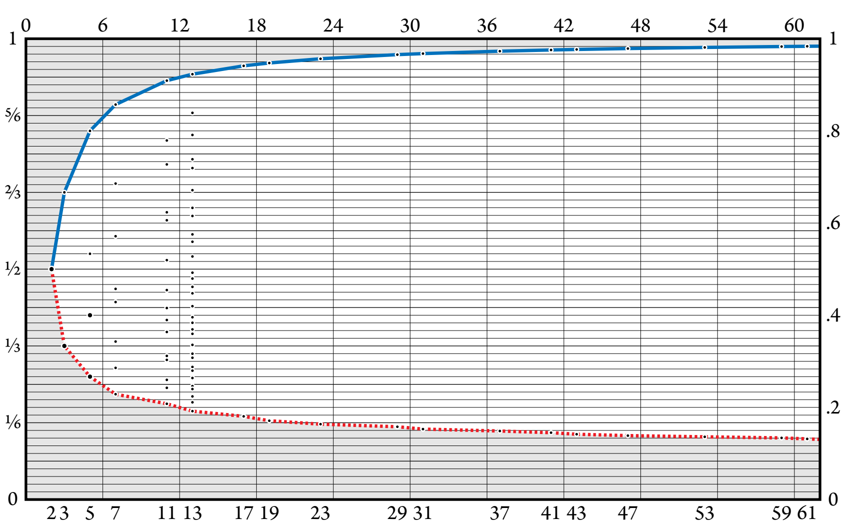


Figure A1. A plot to scale with $\phi(\mathrm{r})/\mathrm{r}$ on the vertical axis versus the maximum distinct prime divisor p_{max} on the horizontal axis. The p_{max}-smooth numbers lie along a vertical line at each value of p_{max}. The boundary of minimum values of $\phi(\mathrm{r})/\mathrm{r}$ defined by primorials is indicated by a broken red line. The boundary of maximum values of $\phi(\mathrm{r})/\mathrm{r}$ defined by primes is shown in blue. All other composite numbers r that have p_{max} as the maximum distinct prime divisor inhabit the region between the boundaries. (See Figure 2.1 for detail at $2 \leq \mathrm{p}_{max} \leq 13$.)

Figure A2: Regular Digit Maps for $2 \le r \le 66$

Figure A3: Regular Digit Maps for $r = 2520$

2520

7^0:

5^0, rows 3^δ, columns 2^δ

1	2	4	8	16	32	64	128	256	512	1024	2048
3	6	12	24	48	96	192	384	768	1536		
9	18	36	72	144	288	576	1152	2304			
27	54	108	216	432	864	1728					
81	162	324	648	1296							
243	486	972	1944								
729	1458										
2187											

5^1, rows 3^δ, columns 2^δ

5	10	20	40	80	160	320	640	1280
15	30	60	120	240	480	960	1920	
45	90	180	360	720	1440			
135	270	540	1080	2160				
405	810	1620						
1215	2430							

5^2, rows 3^δ, columns 2^δ

25	50	100	200	400	800	1600
75	150	300	600	1200	2400	
225	450	900	1800			
675	1350					

5^3, rows 3^δ, columns 2^δ

125	250	500	1000	2000
375	750	1500		
1125	2250			

5^4, rows 3^δ, columns 2^δ

625	1250	2500
1875	750	

7^1:

5^0, rows 3^δ, columns 2^δ

7	14	28	56	112	224	448	896	1792
21	42	84	168	336	672	1344		
63	126	252	504	1008	2016			
189	378	756	1512					
567	1134	2268						
1701								

5^1, rows 3^δ, columns 2^δ

35	70	140	280	560	1120	2240
105	210	420	840	1680		
315	630	1260	2520			
945	1890					

5^2, rows 3^δ, columns 2^δ

175	350	700	1400
525	1050	2100	
1575			

5^3, rows 3^δ, columns 2^δ

875	1750

7^2:

5^0, rows 3^δ, columns 2^δ

49	98	196	392	784	1568
147	294	588	1176	2352	
441	882	1764			
1323					

5^1, rows 3^δ, columns 2^δ

245	490	980	1960
735	1470		
2205			

5^2, rows 3^δ, columns 2^δ

1225	2450

7^3:

5^0, rows 3^δ, columns 2^δ

343	686	1372
1029	2058	

5^1, rows 3^δ, columns 2^δ

1715

7^4:

5^0, rows 3^δ, columns 2^δ

2401

Figure A4: Semitotative Maps for $2 \le r \le 60$

8	2
3	6

9	3
2	6

10	2
3	6

12	2
5	10

14	2	4
3	6	12
5	10	

15	3	5
2	6	10
4	12	

16	2	4
3	6	12
5	10	
7	14	

18	2	3
5	10	15
7	14	

20	2	4	5
3	6	12	15
7	14		
9	18		

21	3	7
2	6	14
4	12	
5	15	
8	18	

22	2	4
3	6	12
5	10	20
7	14	
9	18	

24	2	3	4
5	10	15	20
7	14	21	
11	22		

25	5
2	10
3	15
4	20

26	2	4	8
3	6	12	24
5	10	20	
7	14		
9	18		
11	22		

27	3	9
2	6	18
4	10	
5	15	
7	21	
8	24	

28	2	4	7	8
3	6	12	21	24
5	10	20		
9	18			
11	22			
13	26			

30	2	3	4
7	14	21	28
11	22		
13	26		

32	2	4	8
3	6	12	24
5	10	20	
7	14	28	
9	18		
11	22		
13	26		
15	30		

33	3	9	11
2	6	18	22
4	12		
5	15		
7	21		
8	24		
10	30		

34	2	4	8
3	6	12	24
5	10	20	
7	14	28	
9	18		
11	22		
13	26		
15	30		

35	5	7
2	10	14
3	15	21
4	20	28
6	30	

36	2	3	4	6
5	10	15	20	30
7	14	21	28	
11	22	33		
13	26			
17	34			

38	2	4	8
3	6	12	24
5	10	20	
7	14	28	
9	18	36	
11	22		
13	26		
15	30		
17	34		

39	3	9	13
2	6	18	22
4	12	36	
5	15		
7	21		
8	24		
10	30		
11	33		

40	2	4	5	8	10
3	6	12	22	24	30
7	14	28	35		
9	18	36			
11	22				
13	26				
17	34				
19	38				

42	2	3	4	6	7	8
5	10	15	20	30	35	40
11	22	28				
13	26	39				
17	34					
19	38					

44	2	4	8	11
3	10	12	24	33
5	22	20	40	
7	26	28		
9	34	36		
13	38			
17	34			
19	38			
21	42			

45	3	5	9	15
2	6	10	18	30
4	12	20	36	
7	21	35		
8	24	40		
11	33			
13	39			
14	42			

46	2	4	8
3	6	12	24
5	10	20	40
7	14	28	
9	18	36	
11	22	44	
13	26		
15	30		
17	34		
19	38		
21	42		

48	2	3	4	6	8	9
5	10	15	20	30	40	45
7	14	21	28	42		
11	22	33	44			
13	26	39				
17	34					
19	38					
23	46					

49	7
2	14
3	21
4	28
5	35
6	42

50	2	4	5	8	10	16
3	6	12	15	24	30	48
7	14	28	35			
9	18	36	45			
11	22	44				
13	26					
17	34					
19	36					
21	42					
23	46					

51	3	9	17
2	6	18	34
4	12	36	
5	15	45	
7	21		
8	24		
10	30		
11	33		
13	39		
16	48		

52	2	4	8	13	16
3	6	12	24	39	48
5	10	20	40		
7	14	28			
9	18	36			
11	22	44			
15	30				
17	34				
19	38				
21	42				
23	46				
25	50				

54	2	3	4	6	8	9
5	10	15	20	30	40	45
7	14	21	28	42		
11	22	33	44			
13	26	39	52			
17	34	51				
19	38					
23	46					
25	50					

55	5	11	25
2	10	22	50
3	15	33	
4	20	44	
6	30		
7	35		
8	40		
9	45		

56	2	4	7	8	14	16
3	6	12	21	24	42	48
5	10	20	35	40		
9	18	36				
11	22	44				
13	26	52				
17	34	51				
19	38					
23	46					
25	50					
27	54					

Figure A4 continued.

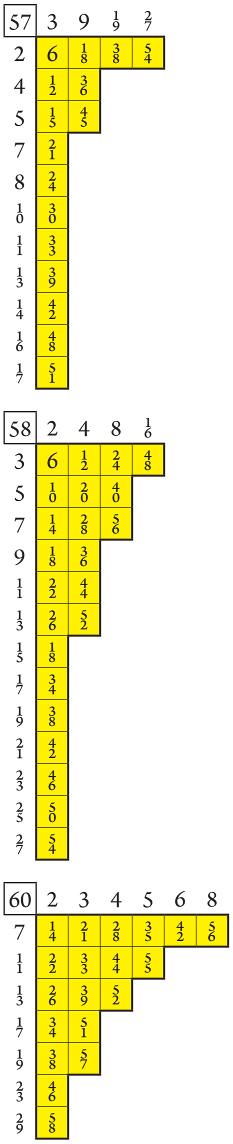
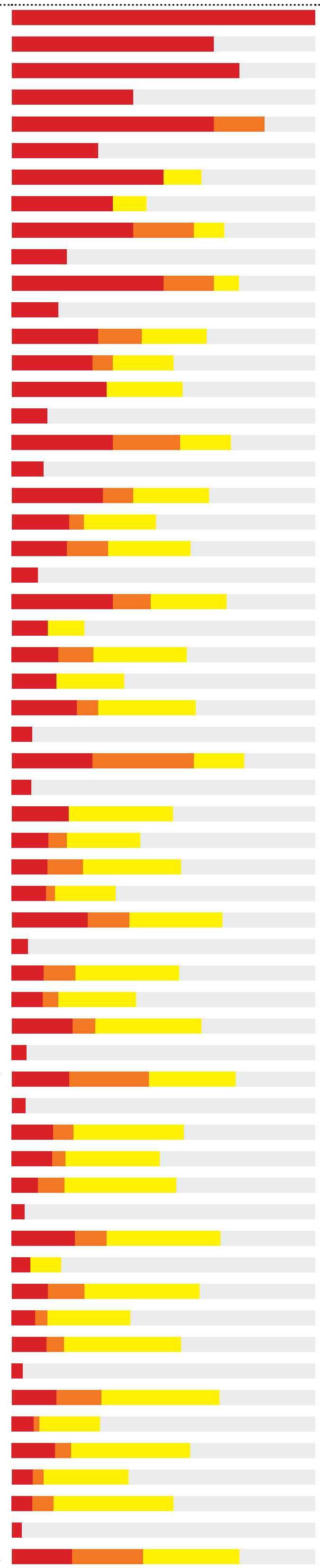
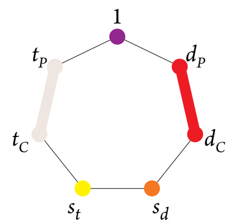


Table A5: Digit Spectra for $2 \le r \le 60$



r	PF	Sparkline	G	D	S_d	S_t	T	N
2	2		2	2	·	·	1	·
3	3		2	2	·	·	2	·
4	2^2		3	3	·	·	2	·
5	5		2	2	·	·	4	·
6	$2 \cdot 3$		5	4	1	·	2	1
7	7		2	2	·	·	6	·
8	2^3		4	4	·	1	4	1
9	3^2		3	3	·	1	6	1
10	$2 \cdot 5$		6	4	2	1	4	3
11	11		2	2	·	·	10	·
12	$2^2 \cdot 3$		8	6	2	1	4	3
13	11		2	2	·	·	12	·
14	$2 \cdot 7$		6	4	2	3	6	5
15	$3 \cdot 5$		5	4	1	3	8	4
16	2^4		5	5	·	4	8	4
17	17		2	2	·	·	16	·
18	$2 \cdot 3^2$		10	6	4	3	6	7
19	19		2	2	·	·	18	·
20	$2^2 \cdot 5$		8	6	2	5	8	7
21	$3 \cdot 7$		5	4	1	5	12	6
22	$2 \cdot 11$		7	4	3	6	10	8
23	23		2	2	·	·	22	·
24	$2^3 \cdot 3$		11	8	3	6	8	9
25	5^2		3	3	·	3	20	3
26	$2 \cdot 13$		7	4	3	8	12	11
27	3^3		4	4	·	6	18	6
28	$2^2 \cdot 7$		8	6	2	9	12	11
29	29		2	2	·	·	28	·
30	$2 \cdot 3 \cdot 5$		18	8	10	5	8	15
31	31		2	2	·	·	30	·
32	2^5		6	6	·	11	16	11
33	$3 \cdot 11$		6	4	2	8	20	10
34	$2 \cdot 17$		8	4	4	11	16	15
35	$5 \cdot 7$		5	4	1	7	24	8
36	$2^2 \cdot 3^2$		14	9	5	11	12	16
37	37		2	2	·	·	36	·
38	$2 \cdot 19$		8	4	4	13	18	17
39	$3 \cdot 13$		6	4	2	10	24	12
40	$2^3 \cdot 5$		11	8	3	14	16	17
41	41		2	2	·	·	40	·
42	$2 \cdot 3 \cdot 7$		19	8	11	12	12	23
43	43		2	2	·	·	42	·
44	$2^2 \cdot 11$		9	6	3	15	20	18
45	$3^2 \cdot 5$		8	6	2	14	24	16
46	$2 \cdot 23$		8	4	4	17	22	21
47	47		2	2	·	·	46	·
48	$2^4 \cdot 3$		15	10	5	18	16	23
49	7^2		3	3	·	5	42	5
50	$2 \cdot 5^2$		12	6	6	19	20	25
51	$3 \cdot 17$		6	4	2	14	32	15
52	$2^2 \cdot 13$		9	6	3	20	24	23
53	53		2	2	·	·	52	·
54	$2 \cdot 3^3$		16	8	8	23	18	31
55	$5 \cdot 11$		5	4	1	11	40	12
56	$2^3 \cdot 7$		11	8	3	22	24	25
57	$3 \cdot 19$		6	4	2	16	36	18
58	$2 \cdot 29$		8	4	4	23	28	27
59	59		2	2	·	·	58	·
60	$2^2 \cdot 3 \cdot 5$		26	12	14	19	16	35

Table A6: Digit Maps for $2 \le r \le 60$



Legend

1 **Unit**, $(1 \mid r) \wedge 1 \perp r$

d **Divisor**, primes $d_p \mid r$, and composites $d_c \mid r$, the set of divisors of $r = \mathcal{D}_r$

s_d **Semidivisor**, a composite $s_d \mid kr$, where its divisors $d_n \in \mathcal{D}_r$

s_t **Semitotative**, a composite $s_t \mid kr$, where at least one of its divisors $d_n \notin \mathcal{D}_r$

t **Totative**, primes $t_p \perp r$, and composites $t_c \perp r$, the set of totatives of $r = \mathcal{T}_r$

r	n																																																											
2	0	1
3	0	1	2
4	0	1	2	3
5	0	1	2	3	4
6	0	1	2	3	4	5
7	0	1	2	3	4	5	6
8	0	1	2	3	4	5	6	7
9	0	1	2	3	4	5	6	7	8
10	0	1	2	3	4	5	6	7	8	9
11	0	1	2	3	4	5	6	7	8	9	10
12	0	1	2	3	4	5	6	7	8	9	10	11
13	0	1	2	3	4	5	6	7	8	9	10	11	12
14	0	1	2	3	4	5	6	7	8	9	10	11	12	13
15	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
16	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
17	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
18	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
19	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
20	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
21	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
22	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
23	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
24	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
25	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
26	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
27	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
28	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
29	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
30	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
31	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
32	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
33	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
34	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
35	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
36	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
37	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
38	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
39	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38
40	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
41	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
42	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41
43	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
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